IEEE Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology— Random Instabilities

Sponsor

IEEE Standards Coordinating Committee 27 on Time and Frequency

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IEEE-SA Standards Board

Abstract: Methods of describing random instabilities of importance to frequency and time metrology is covered in this standard. Quantities covered include frequency, amplitude, and phase instabilities; spectral densities of frequency, amplitude, and phase fluctuations; and time-domain variances of frequency fluctuations. In addition, recommendations are made for the reporting of measurements of frequency, amplitude and phase instabilities, especially as regards the recording of experimental parameters, experimental conditions, and calculation techniques.
Keywords: AM noise, amplitude instability, FM noise , frequency domain, frequency instability, frequency metrology, frequency modulation, noise, phase instability, phase modulation, phase noise, PM noise, time domain, time metrology

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Introduction

(This introduction is not part of IEEE Std 1139-1999, IEEE Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology—Random Instabilities.)

Techniques to characterize and to measure the frequency, phase, and amplitude instabilities in frequency and time devices and in received radio signals are of fundamental importance to all manufacturers and users of frequency and time technology.

In 1964, the Standards Coordinating Committee 14 and, in 1966, the Technical Committee on Frequency and Time were formed to prepare an IEEE standard on frequency stability. In 1969, these committees completed a document proposing definitions for measures of frequency and phase stabilities (Barnes [B12][a]). In 1988, an updated IEEE standard on frequency stability was published by the IEEE as IEEE Std 1139-1988, Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology. The recommended measures of instabilities in frequency generators have gained acceptance among frequency and time users throughout the world.

This standard is a revision of IEEE Std 1139-1988, which had been prepared by a previous Standards Coordinating Committee 14 consisting of Helmut Hellwig, Chairman; David Allan; Peter Kartaschoff; Jacques Vanier; John Vig; Gernot M. R. Winkler; and Nicholas F. Yannoni. Some clauses of the 1988 standard remain unchanged.

Most of the major manufacturers now specify instability characteristics of their standards in terms of the recommended measures. This standard thus defines and formalizes the general practice.

This revision was prepared by the Standards Coordinating Committee 27. At the time this standard was approved, the membership of the Standards Coordinating Committee 27 was as follows:

The authors thank C. A. Greenhall, D. A. Howe, A. Lepek, M. A. Lombardi, and M. A. Weiss for helpful discussions.

[a]The numbers in brackets correspond to those of the bibliography in Annex E.

When the IEEE-SA Standards Board approved this standard on 26 June 1999, it had the following membership:

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IEEE Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology—Random Instabilities

1. Overview

1.1 Scope

This standard covers methods of describing random instabilities of importance to frequency and time metrology. Quantities covered include frequency, amplitude, and phase instabilities; spectral densities of frequency, amplitude, and phase fluctuations; and time-domain measures of frequency fluctuations. In addition, recommendations are made for the reporting of measurements of frequency, amplitude and phase instabilities, especially as regards the recording of experimental parameters, experimental conditions, and calculation techniques. Basic concepts and definitions, time prediction, confidence limits when estimating the variance from a finite data set, and confidence limits for frequency domain measures are covered in the annexes. The annexes also cover translation between the frequency-domain and time-domain measures, examples on how to calculate the time-domain measures of frequency fluctuations, and an extensive bibliography of the relevant literature. Systematic instabilities, such as environmental effects and aging, are discussed in IEEE Std 1193-1994 [B40][1].

2. Summary of definitions

The standard definitions given in the main body of the text and the annexes are listed here in narrative form. If ambiguities are created between the narrative definition given here and the mathematical equation given in the text, the text has priority.

2.1 amplitude deviation $\varepsilon(t)$: Instantaneous amplitude departure from a nominal amplitude.

2.2 amplitude instability $S_a(f)$: One-sided spectral density of the fractional amplitude deviation.

[1]The numbers in brackets correspond to those of the bibliography in Annex E.

2.3 confidence limit: The uncertainty associated with the estimate of a time- or frequency-domain instability measure from a finite number of measurements.

2.4 frequency deviation $y(t)$: Instantaneous, normalized, or fractional frequency departure from a nominal frequency.

2.5 frequency instability $S_y(f)$: One-sided spectral density of the fractional frequency deviation.

2.6 phase deviation $\phi(t)$: Instantaneous phase departure from a nominal phase.

2.7 phase instability $S_\phi(f)$: One-sided spectral density of the phase deviation.

2.8 phase noise $\mathcal{L}(f)$: One-half of the phase instability $S_\phi(f)$, as defined in 1.7.

2.9 time deviation $x(t)$: Instantaneous time departure from a nominal time.

2.10 time instability $S_x(f)$: One-sided spectral density of the time deviation.

2.11 time interval error (TIE): The variation of the time difference between a real clock and an ideal uniform time scale following a time period t after perfect synchronization.

2.12 two-sample deviation $\sigma_y(\tau)$: Also called the Allan deviation; the square root of the two-sample variance, as defined in 2.13.

2.13 two-sample variance $\sigma_y^2(\tau)$: Also called the Allan variance; time average over the sum of the squares of the differences between successive readings of the normalized frequency departure sampled over the sampling time τ, under the assumption that there is no dead time between the normalized frequency departure samples.

3. Standards for characterizing or reporting measurements of frequency, amplitude, and phase instabilities

The standard measure for characterizing frequency and phase instabilities in the frequency domain is the phase noise, $\mathcal{L}(f)$ (pronounced "script-ell of f"), defined as one half of the double-sideband spectral density of phase fluctuations,

$$\mathcal{L}(f) \equiv \frac{1}{2} S_\phi(f) .$$

When expressed in decibels, the units of $\mathcal{L}(f)$ are dBc/Hz (dB below the carrier in a 1 Hz bandwidth). A device shall be characterized by a plot of $\mathcal{L}(f)$ versus Fourier frequency f. In some applications, providing $\mathcal{L}(f)$ versus discrete values of Fourier frequency is sufficient. (See Annexes A and B for further discussion). The standard measure for characterizing amplitude instability in the frequency domain is one half of the double-sideband spectral density of the fractional amplitude fluctuations, $1/2S_a(f)$. When expressed in decibels the units of $1/2S_a(f)$ are dBc/Hz (dB below the carrier in a 1 Hz bandwidth). See Annex A for a detailed discussion on spectral densities.

In the time domain, the standard measure of frequency and phase instabilities is the fully overlapped Allan deviation $\sigma_y(\tau)$. A device shall be characterized by a plot of $\sigma_y(\tau)$ versus sampling time τ. In some cases, providing discrete values of $\sigma_y(\tau)$ versus sampling time τ is sufficient. (See Annex A and Annex B for further discussion). The measurement system bandwidth (f_h) and the total measurement time shall be indicated.

In addition, the provisions in 3.1 and 3.2 are recommended when reporting measurements on frequency and phase instabilities.

3.1 Nonrandom phenomena should be recognized

Examples of nonrandom phenomena include

a) Any observed time dependence of the statistical measures should be stated.
b) The method of modeling systematic behavior should be specified (e.g., an estimate of the linear frequency drift was obtained from the coefficients of a linear least-squares regression to M frequency measurements, each with a specified averaging or sample time t and measurement bandwidth f_h).
c) The environmental sensitivities should be stated (e.g., the dependence of frequency and/or phase on temperature, magnetic field, barometric pressure, and vibration).

3.2 Relevant measurement or specification parameters should be given

Relevant measurement or specification parameters include

a) The nominal signal frequency ν_o.
b) The method of measurements.
c) The measurement system bandwidth f_h and the corresponding low-pass filter response.
d) The total measurement time (data sample) and the number of measurements N.
e) The characteristics of the reference signal (equal noise or much lower noise assumed).
f) For sample averaging times that exceed 10% of the total measurement time, $\hat{\sigma}_{y,\,\mathrm{TOTAL}}(\tau)$ should be used to estimate $\sigma_y(\tau)$ (see Howe [B35] and [B36] and Howe and Greenhall [B37]).
g) The calculation techniques [e.g., details of the window function when estimating power spectral densities from time-domain data or the assumptions about effects of dead time when estimating the two-sample deviation $\sigma_y(t)$].
h) The confidence of the estimate (or uncertainty) and its statistical probability (e.g., 1σ for 68%, 2σ for 95%). See Annex D.
i) The environment during measurement.

Annex A

(informative)

Measures of frequency, amplitude, and phase instabilities

A.1 Introduction

The instantaneous output voltage of a precision oscillator can be expressed as

$$V(t) = (V_o + \varepsilon(t)) \sin (2\pi\nu_o t + \phi(t)), \tag{A.1}$$

where

- V_o is the nominal peak voltage amplitude,
- $\varepsilon(t)$ is the deviation from the nominal amplitude,
- ν_o is the nominal frequency,
- $\phi(t)$ is the phase deviation from the nominal phase $2\pi\nu_o t$.

Figure A.1[2] illustrates a signal with frequency, amplitude, and phase instabilities. As shown, frequency instability is the result of fluctuations in the period of oscillation. Fluctuations in the phase result in instability of the zero crossing. Fluctuations in the peak value of the signal (V_{peak}) result in amplitude instability.

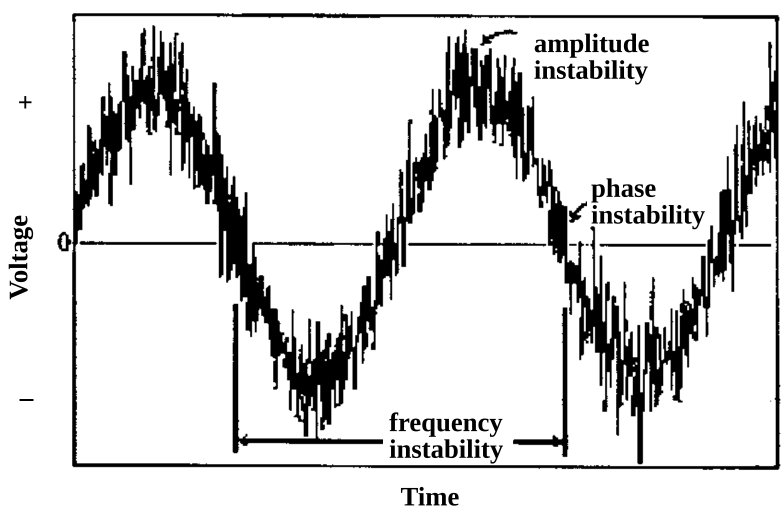

Figure A.1—Instantaneous output voltage of an oscillator

Frequency instability of a precision oscillator is defined in terms of the instantaneous, normalized frequency deviation, $y(t)$, as follows

$$y(t) \equiv \frac{\nu(t) - \nu_o}{\nu_o} = \frac{\dot{\phi}(t)}{2\pi\nu_o}, \tag{A.2}$$

[2]In the signal shown in Figure A.1, the frequency components of the noise are higher than the carrier frequency. The higher frequency noise components are used for illustration purposes only. In general, this standard applies to the frequency components of amplitude, phase, and frequncy instabilities that are lower in frequency than the carrier frequency.

where $\nu(t)$ is the instantaneous frequency (time derivative of the phase divided by 2π), and

$$\dot{\phi}(t) = \frac{d\phi(t)}{dt}. \quad \text{(A.3)}$$

Amplitude instability is defined in terms of the instantaneous, normalized amplitude deviation

$$a(t) \equiv \varepsilon(t)/V_o. \quad \text{(A.4)}$$

Phase instability, defined in terms of the instantaneous phase deviation $\phi(t)$, can also be expressed in units of time, as

$$x(t) = \phi(t)/2\pi\nu_o. \quad \text{(A.5)}$$

With this definition, the instantaneous, normalized frequency deviation is

$$y(t) = dx(t)/dt. \quad \text{(A.6)}$$

Other random phenomena observed in certain oscillators are frequency jumps, that is, discontinuities in the frequency of oscillation. These phenomena are not repetitive or well understood and thus cannot be characterized by standard statistical methods.

A.2 Frequency domain

In the frequency domain, frequency, amplitude, and phase instabilities can be defined or measured by one-sided spectral densities.

The measure of frequency instability is the spectral density of fractional frequency fluctuations, $S_y(f)$, given by

$$S_y(f) = y^2(f) \cdot \frac{1}{BW}, \quad \text{(A.7)}$$

where

$y(f)$ is the root mean squared (rms) fractional frequency deviation as a function of Fourier frequency,
BW is the measurement system bandwidth in Hz.

The units of $S_y(f)$ are 1/Hz.

The measure of amplitude instability is the spectral density of fractional amplitude fluctuations, $S_a(f)$, given by

$$S_a(f) = \left(\frac{\varepsilon(f)}{V_o}\right)^2 \frac{1}{BW}. \quad \text{(A.8)}$$

The units of $S_a(f)$ are 1/Hz.

Phase instability can be characterized by the spectral density of phase fluctuations, $S_\phi(f)$, given by

$$S_\phi(f) = \phi^2(f)\frac{1}{BW}. \tag{A.9}$$

The units of $S_\phi(f)$ are rad^2/Hz.

These spectral densities, $S_y(f)$, $S_a(f)$, and $S_\phi(f)$, are one-sided since the Fourier frequency f ranges from 0 to ∞; nevertheless, they include fluctuations from both the upper and the lower sidebands of the carrier.

$S_\phi(f)$ is the quantity that is generally measured in frequency metrology; however, $\mathcal{L}(f)$ has become the prevailing measure of phase noise among manufacturers and users of frequency standards. According to the old definition (see Kartaschoff [B42]), $\mathcal{L}(f)$ is the ratio of the power in one sideband due to phase modulation by noise (for a 1 Hz bandwidth) to the total signal power (carrier plus sidebands); that is,

$$\mathcal{L}(f) = \frac{\text{power density in one phase noise modulation sideband, per Hz}}{\text{total signal power}}. \tag{A.10}$$

Usually $\mathcal{L}(f)$ is expressed in decibels (dB) as 10 log ($\mathcal{L}(f)$), and its units are dB below the carrier in a 1 Hz bandwidth, generally abbreviated as dBc/Hz.

The old definition of $\mathcal{L}(f)$ is related to $S_\phi(f)$ by

$$\mathcal{L}(f) \cong \frac{S_\phi(f)}{2}. \tag{A.11}$$

This definition breaks down when the mean squared phase deviation, $\langle\phi^2(t)\rangle$ = the integral of $S_\phi(f)$ from f to ∞, exceeds about 0.1 rad^2. To circumvent this difficulty, $\mathcal{L}(f)$ is redefined as

$$\mathcal{L}(f) \equiv \frac{S_\phi(f)}{2}. \tag{A.12}$$

This redefinition is intended to avoid difficulties in the use of $\mathcal{L}(f)$ in situations where the small angle approximation is not valid. $\mathcal{L}(f)$, as defined by Equation (A.12), should be designated as the standard measure of phase instability in the frequency domain. The reasons are that

— It can always be measured unambiguously, and
— It conforms to the prevailing usage.

Phase instability has sometimes been expressed in units of time by $S_x(f)$, the one-sided spectral density of the phase fluctuations expressed in units of time ($x(t)$),

$$S_x(f) = x^2(f)\frac{1}{BW}. \tag{A.13}$$

From Equation (A.5), $S_\phi(f)$ and $S_x(f)$ are related by

$$S_x(f) = \frac{1}{(2\pi\nu_o)^2}S_\phi(f). \tag{A.14}$$

Since phase and frequency are directly related, that is, angular frequency is the time derivative of the phase, spectral densities of frequency and phase instabilities are also related:

$$S_y(f) = \frac{f^2}{\nu_o^2} S_\phi(f) . \tag{A.15}$$

Other quantities related to phase instability are phase jitter and wander. Phase jitter is the integral of $S_\phi(f)$ over the Fourier frequencies of the application (usually above 10 Hz). When reporting phase jitter, the range of Fourier frequencies should be specified. Wander refers to the integral of $S_\phi(f)$ at Fourier frequencies below 10 Hz. It is possible to convert $S_\phi(f)$ to phase jitter by using the relation

$$\phi_{\text{jitter}}^2 = \int_{f_1}^{f_2} S_\phi(f) df , \tag{A.16}$$

where ϕ_{jitter}^2 refers to the phase jitter. It is not possible to obtain $S_\phi(f)$ from phase jitter, unless the shape of $S_\phi(f)$ is known.

A.3 Time domain

In the time domain, sequential average frequency instabilities are defined by a two-sample deviation $\sigma_y(\tau)$, also called the Allan deviation, which is the square root of a two-sample variance $\sigma_y^2(\tau)$, also called the Allan variance. This variance $\sigma_y^2(\tau)$ has no account of dead time between adjacent frequency samples. (Dead time refers to the time between time-ordered data sets when no measurement of frequency is taken.) For the sampling interval τ

$$\sigma_y(\tau) = \left[\frac{1}{2}\langle[\bar{y}(t+\tau) - \bar{y}(t)]^2\rangle\right]^{1/2} = \left[\frac{1}{2}\langle[\bar{y}_{k+1} - \bar{y}_k]^2\rangle\right]^{1/2} \tag{A.17}$$

where

$$\bar{y}_k = \frac{1}{\tau}\int_{t_k}^{t_k+\tau} y(t)dt = \frac{x(t_k+\tau) - x(t_k)}{\tau} = \frac{x_{k+1} - x_k}{\tau} . \tag{A.18}$$

The symbol < > denotes an infinite time average, and τ is the sampling interval. In practice, the requirement of infinite time average is never fulfilled, and the Allan deviation is estimated by

$$\sigma_y(\tau) \cong \left[\frac{1}{2(M-1)}\sum_{k=1}^{M-1}(\bar{y}_{k+1} - \bar{y}_k)^2\right]^{1/2} , \tag{A.19}$$

where M is the number of frequency measurements.

The Allan deviation can also be expressed in terms of time difference (or time residual) measurements by combining Equation (A.18) and Equation (A.19):

$$\sigma_y(\tau) \cong \left[\frac{1}{2(N-2)\tau^2}\sum_{k=1}^{N-2}(x_{k+2} - 2x_{k+1} + x_k)^2\right]^{1/2} , \tag{A.20}$$

where

x_k, x_{k+1}, and x_{k+2}	are time residual measurements made at t_k, $t_{k+1} = t_k+\tau$, and $t_{k+2} = t_k+2\tau$,
k	is 1, 2, 3, ...,
N	is the number of time measurements,
$1/\tau$	is the nominal fixed sampling rate that gives zero dead time between frequency measurements.

Residual implies the consistent, systematic effects, such as frequency drift, have been removed (see A.3).

If there is dead time between the frequency departure measurements and it is ignored in the computation of $\sigma_y(\tau)$, resulting instability values will be biased (except for white frequency noise). Some of the biases have been studied and some correction tables published (see Barnes [B11], Barnes and Allan [B15], and Lesage [B49]). Therefore, the term $\sigma_y(\tau)$ shall not be used to describe such biased measurements without stating the bias together with $\sigma_y(\tau)$. The unbiased $\sigma_y(\tau)$ can be calculated from the biased values, using information in the references. Considering that $\{x_k\}$ can be routinely measured, it is preferred that $\{x_k\}$ is used to compute $\sigma_y(\tau)$ since the problem of dead time is solved.

If the initial sampling rate is specified as $1/\tau_0$, then, in general, an estimate of $\sigma_y(\tau)$ with better confidence may be obtained using what is called overlapping estimates. This estimate is obtained by computing

$$\sigma_y(\tau) = \left[\frac{1}{2(N-2m)\tau^2}\sum_{k=1}^{N-2m}(x_{k+2m}-2x_{k+m}+x_k)^2\right]^{1/2}, \tag{A.21}$$

where

N	is the number of original time residual measurements spaced by τ_o ($N = M + 1$, where M is the number of original frequency measurements of sample time τ_o),
τ	is $m\tau_o$,
m	is 1, 2, 3, ...

Examples of overlapped $\sigma_y(\tau)$ estimates are given in Annex C.

Equation (A.21) shows that $\sigma_y(\tau)$ acts like a second-difference operator on the time deviation residuals usually providing a stationary measure of the stochastic behavior even for nonstationary processes. An efficient spacing of τ values in a plot of $\log[\sigma_y(\tau)]$ versus $\log[\tau]$ sets $m = 2^p$, where $p = 0, 1, 2, 3, ...$

When differentiating between white and flicker phase modulation noise is desirable, a modified deviation, denoted as Mod $\sigma_y(\tau)$, may be used to characterize frequency instabilities (see Allan and Barnes [B6] and Stein [B64]). Unlike $\sigma_y(\tau)$, Mod $\sigma_y(\tau)$ has the property of yielding different dependence on τ for white phase noise and flicker phase noise; the dependencies are $\tau^{-3/2}$ and τ^{-1}, respectively. [The dependence for $\sigma_y(\tau)$ is τ^{-1} for both white and flicker phase noise.] Another advantage is that Mod $\sigma_y(\tau)$ averages wideband phase noise faster than τ^{-1}. Mod $\sigma_y(\tau)$ is defined as

$$\text{Mod } \sigma_y(\tau) = \left\{\frac{1}{2\tau^2 m^2(N-3m+1)}\sum_{j=1}^{N-3m+1}\left[\sum_{i=j}^{m+j-1}(x_{i+2m}-2x_{i+m}+x_i)\right]^2\right\}^{1/2} \tag{A.22}$$

For examples of $\sigma_y(\tau)$ and Mod $\sigma_y(\tau)$, see Annex C.

A measure that is often used in time transfer systems, such as the global positioning system (GPS), is $\sigma_x(\tau)$. $\sigma_x(\tau)$ is defined as

$$\sigma_x(\tau) = \frac{\tau}{\sqrt{3}}\mathrm{Mod}\ \sigma_y(\tau)\,. \tag{A.23}$$

This quantity is useful when white and flicker of phase modulation noise dominate a synchronization system.

Another approach to distinguish different noise types is to use multivariance analysis (see Vernotte [B68]). By using several variances to analyze the same data, it is possible to estimate the coefficients for the five noise types. For a description of noise types, see Annex B.

At long averaging times, when the sample averaging time exceeds 10% of the total measurement time, the Allan deviation has potential errors and a bias related to its insensitivity to odd (antisymmetric) noise processes in $x(t)$ [odd about the midpoint of the $x(t)$ data or even in terms of average $\bar{y}_k$]. This insensitivity to odd noise processes is illustrated in Figure A.2. Part (a) of Figure A.2 shows three phase samples of a noise process, which is odd about x_2. The calculated fractional frequency deviations according to Equation (A.19) are shown in Part (b) of Figure A.2. Since $\bar{y}_1$ and $\bar{y}_2$ are equal, contributions due to this noise process will not show up in $\sigma_y(\tau)$.

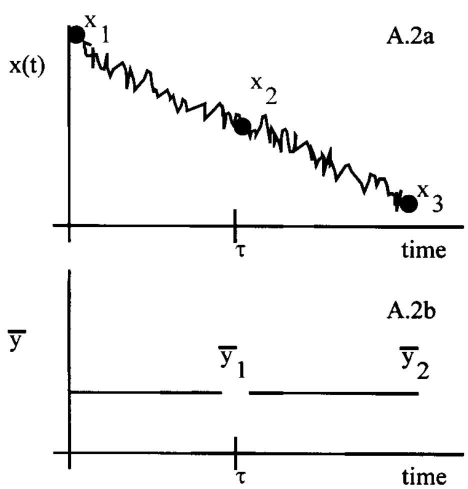

Figure A.2—Odd noise process about x_2

For this reason, when τ exceeds 10% of the data sample, using the total deviation $\hat{\sigma}_{y,\,\mathrm{TOTAL}}(\tau)$ to estimate $\sigma_y(\tau)$ is recommended. $\hat{\sigma}_{y,\,\mathrm{TOTAL}}(\tau)$ extends the x_k sequence at both ends by reflection about the endpoints to provide a better estimate of frequency stability. The advantages of $\hat{\sigma}_{y,\,\mathrm{TOTAL}}(\tau)$ are outlined in several references (see Howe [B35] and [B36] and Howe and Greenhall [B37]). To define it, let $x_1, ..., x_N$ ($N \geq 5$) be the time-residual data, sampled with time period τ_0. The maximum value of τ considered here is $m_{\max}\tau_0$, where $m_{\max}$ is the integer part of $(N-1)/2$. Construct a new, longer sequence $\{x'_k\}$ as follows:

- — For $k = 1$ to N let $x'_k = x_k$;
- — On the left define $x'_0 = 2x_1 - x_2$, $x'_{-1} = 2x_1 - x_3$, ..., $x'_{2-m_{\max}} = 2x_1 - x_{m_{\max}}$;
- — On the right define $x'_{N+1} = 2x_N - x_{N-1}$, $x'_{N+2} = 2x_N - x_{N-2}$, ..., $x'_{N+m_{\max}-1} = 2x_N - x_{N-m_{\max}+1}$.

For $\tau = m\tau_0$ let

$$\hat{\sigma}_{y,\mathrm{TOTAL}}(\tau) = \left(\frac{1}{2\tau^2(N-2)}\sum_{k=2}^{N-1}[x'_{k-m} - 2x'_k + x'_{k+m}]^2\right)^{1/2}. \tag{A.24}$$

$\hat{\sigma}_{y,\,\mathrm{TOTAL}}(\tau)$ can also be represented in terms of extended fractional frequency fluctuation averages as

$$\hat{\sigma}_{y,\mathrm{TOTAL}}(\tau) = \left[\frac{1}{2(N-2)}\sum_{k=2}^{N-1}(\bar{y}'_k - \bar{y}'_{k-m})^2\right]^{1/2}, \tag{A.25}$$

where $\bar{y}'_k = (x'_{k+m} - x'_k)/\tau$.

A.4 Systematic instabilities

The long-term frequency change of a source is called frequency drift. Drift includes frequency changes caused by changes in the components of the oscillator, in addition to sensitivities to the oscillator's changing environment and changes caused by load and power supply changes (see Vig and Meeker [B71]).

The frequency aging of an oscillator refers to the change in the frequency of oscillation caused by changes in the components of the oscillator, either in the resonant unit or in the accompanying electronics. Aging differs from drift in that it does not include frequency changes due to changes in the environment, such as temperature changes. Aging is thus a measure of the long-term stability of the oscillator, independent of its environment. The frequency aging of a source (positive or negative) is typically maximum immediately after turn-on.

Aging can be specified by the normalized rate of change in frequency at a specified time after turn-on (e.g., 1×10^{-10} per day after 30 days) or by the total normalized change in frequency in a period of time (e.g., 1×10^{-8} per month) (see Vig and Meeker [B71]).

A.5 Clock-time prediction

The variation of the time difference between a real clock and an ideal uniform time scale, also known as time interval error TIE, observed over a time interval starting at time t_0 and ending at $t_0 + t$ is defined as

$$\mathrm{TIE}(t) = x(t_0 + t) - x(t_0) = \int_{t_0}^{t_0+t} y(t')dt'. \tag{A.26}$$

For fairly simple models, regression analysis can provide efficient estimates of the TIE (see Draper and Smith [B28] and CCIR [B23]). In general, many estimators are possible for any statistical quantity. An efficient and unbiased estimator is preferred. Using the time-domain measure $\sigma_y(\tau)$, the following estimate of the standard deviation (rms) of TIE and its associated systematic departure due to a linear frequency drift (plus its uncertainty) can be used to predict a probable time interval error of a clock synchronized at $t = t_0 = 0$ and left free running thereafter:

$$\mathrm{rms\ TIE_{est}}(t) = t\left(\left[\frac{x(t_0)}{t}\right]^2 + \sigma_{y_0}^2 + \sigma_y^2(\tau = t) + \frac{a^2}{4}t^2\right)^{1/2}, \tag{A.27}$$

where

- $x(t_0)$ is the uncertainty in the initial synchronization,
- σ_{y_0} is the uncertainty in the initial frequency adjustment,
- $\sigma_y(\tau)$ is the two-sample deviation describing the random frequency instability of the clock at $\tau = t$ computed after a linear frequency drift has been removed,
- a is the normalized linear frequency drift per unit of time plus the uncertainty in the drift estimate.

The third term in the brackets provides an optimum and unbiased estimate [under the condition of an optimum (rms) prediction method] in the cases of white noise frequency modulation and/or random walk frequency modulation. The third term is too optimistic, by about a factor of 1.4, for flicker noise frequency modulation and too pessimistic, by about a factor of 3, for white noise phase modulation (see Barnes and Allan [B10] and Allan [B7]).

This estimate is a useful and fairly simple approximation. A more complete error analysis becomes difficult. If carried out, such an analysis needs to include the methods of time prediction, the uncertainties of the clock parameters using the confidence limits of measurements defined below, the detailed clock noise models, systematic effects, etc.

A quantity often used to characterize the stability of clocks in telecommunication systems is the maximum time interval error (MTIE). MTIE is defined as the maximum time difference minus the minimum time difference between a clock and an ideal reference (see Bregni [B22]).

Annex B

(informative)

Power-laws and conversion between frequency and time domain

B.1 Power-law spectral densities

Power-law spectral densities serve as reasonable and accurate models of the random fluctuations in precision oscillators. In practice, these random fluctuations can often be represented by the sum of five such noise processes assumed to be independent, as

$$S_y(f) = \begin{cases} \sum_{\alpha=-2}^{+2} h_\alpha f^\alpha & \text{for } 0 < f < f_h, \\ 0 & \text{for } f \geq f_h, \end{cases} \tag{B.1}$$

where

h_α is constant,
α is integer,
f_h is high-frequency cut-off of an infinitely sharp low pass filter.

High frequency divergence is eliminated by the restrictions on f in this equation. The identification and characterization of the five noise processes are given in Table B.1 and shown in Figure B.1.

Table B.1—Functional characteristics of the independent noise processes used in modeling frequency instability of oscillators

	Slope characteristics of log-log plot				
	Frequency domain		Time domain		
	$S_y(f)$ [a]	$S_\phi(f)$ [b] or $S_x(f)$ [c]	$\sigma_y^2(\tau)$ [d]	$\sigma_y(\tau)$ [e]	Mod $\sigma_y(\tau)$ [f]
Description of noise process	α	β	μ	$\mu/2$	μ'
Random walk frequency modulation	–2	–4	1	1/2	1/2
Flicker frequency modulation	–1	–3	0	0	0
White frequency modulation	0	–2	–1	–1/2	–1/2
Flicker phase modulation	1	–1	–2	–1	–1
White phase modulation	2	0	–2	–1	–3/2

[a] $S_y(f) = \frac{(2\pi f)^2}{(2\pi\nu_0)^2} S_\phi(f) = h_\alpha f^\alpha$

[b] $S_\phi(f) = \nu_0^2 h_\alpha f^{\alpha-2} = \nu_0^2 h_\alpha f^\beta \quad (\beta \equiv \alpha - 2)$

[c] $S_x(f) = \frac{1}{4\pi^2} h_\alpha f^{\alpha-2} = \frac{1}{4\pi^2} h_\alpha f^\beta$

[d] $\sigma_y^2(\tau) \sim |\tau|^\mu$

[e] $\sigma_y(\tau) \sim |\tau|^{\mu/2}$

[f] Mod $\sigma_y(\tau) \sim |\tau|^{\mu'}$

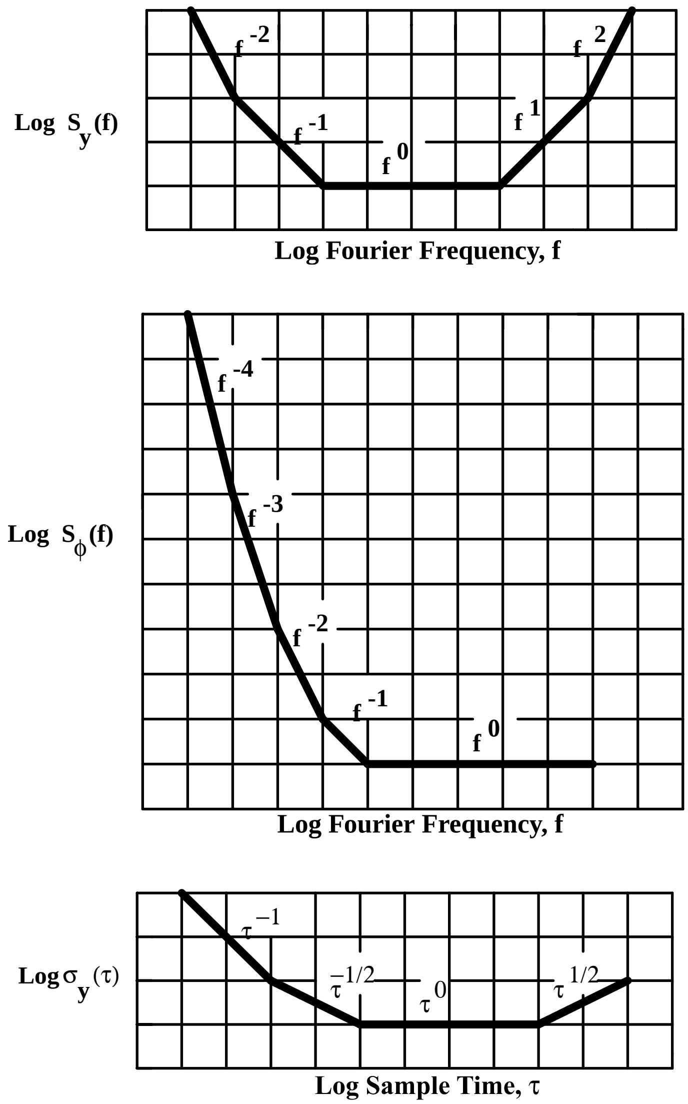

Figure B.1—Slope characteristics of the five independent noise processes

B.2 Conversion between frequency and time domain

The operation of the counter, averaging the frequency for a time τ, may be thought of as a filtering operation. The transfer function $H(f)$ of this equivalent filter is then the Fourier transform of the impulse response of the filter. The time-domain frequency instability is then given by

$$\sigma^2(M, T, \tau) = \int_0^{\infty} S_y(f)|H(f)|^2 df , \tag{B.2}$$

where $S_y(f)$ is the spectral density of frequency fluctuations. $1/T$ is the measurement rate. ($T - \tau$ is the dead time between measurements.) In the case of the two-sample variance, $|H(f)|^2$ is $2(\sin^4 \pi\tau f)/(\pi\tau f)^2$. The two-sample variance can thus be computed from

$$\sigma_y^2(\tau) = 2\int_0^{f_h} S_y(f)\frac{\sin^4(\pi\tau f)}{(\pi\tau f)^2}\, df . \tag{B.3}$$

Specifically, for the power law model given, the time-domain measure also follows a power law:

$$\sigma_y^2(\tau) = h_{-2}\frac{(2\pi)^2}{6}\tau + h_{-1}2\log_e 2 + h_0\frac{1}{2\tau} + h_1\frac{1.038 + 3\log_e 2\pi f_h\tau}{2\pi^2\tau^2} + h_2\frac{3f_h}{(2\pi)^2\tau^2} . \tag{B.4}$$

Equation (B.4) assumes that f_h is the high-frequency cut-off of an infinitely sharp low-pass filter and that $2\pi f_h\tau >> 1$. This equation also implicitly assumes that the random driving mechanism for each term is independent of the others and that the mechanism is valid over all Fourier frequencies. These assumptions may not always be true.

The values of h_α are characteristic models of oscillator frequency noise. For integer values (as often seems to be the case for reasonable models),

$\mu = -\alpha - 1$, for $-3 < \alpha \le 1$,

$\mu \approx -2$ for $\alpha \ge 1$,

where

$\sigma_y{}^2(\tau) \sim \tau^\mu$.

The modified two-sample variance can also be computed from $S_y(f)$ by using

$$\text{Mod } \sigma_y^2(\tau) = \frac{2}{n^4}\int_0^{f_h} S_y(f)\frac{\sin^6(\pi\tau f)}{(\pi\tau_0 f)^2\sin^2(\pi\tau_0 f)}df . \tag{B.5}$$

Table B.2 gives the coefficients of the translation from $S_y(f)$ (frequency domain) to $\sigma_y{}^2(\tau)$ (time domain). In general computation of $S_y(f)$ or related frequency domain measurements from $\sigma_y(\tau)$ or Mod $\sigma_y(\tau)$ are not permitted unless only one power law noise type is present. Nevertheless, when several noise types are present, special analysis can be made on the time-domain data to obtain the coefficients (in the frequency domain) for each power law (see Vernotte [B68]).

The slope characteristics of the five independent noise processes are plotted in the frequency and time domains in Figure B.1 (log-log scale).

Table B.2—Translation of frequency instability measures from spectral densities in frequency domain to variances in time domain for an infinitely sharp low-pass filter with $2\pi f_h\tau >> 1$

Description of noise process	$S_y(f) =$	$S_\phi(f) =$	$\sigma_y^2(\tau) =$ [a]
Random walk frequency modulation	$h_{-2}f^{-2}$	$h_{-2}\nu^2 f^{-4}$	$Ah_{-2}\tau^1$
Flicker frequency modulation	$h_{-1}f^{-1}$	$h_{-1}\nu^2 f^{-3}$	$Bh_{-1}\tau^0$
White frequency modulation	$h_0 f^0$	$h_0\nu^2 f^{-2}$	$Ch_0\tau^{-1}$
Flicker phase modulation	$h_1 f^1$	$h_1\nu^2 f^{-1}$	$Dh_1\tau^{-2}$
White phase modulation	$h_2 f^2$	$h_2\nu^2 f^0$	$Eh_2\tau^{-2}$

[a]

$$A = \frac{2\pi^2}{3}$$

$B = 2\ 1n\ 2$

$C = 1/2$

$$D = \frac{1.038 + 3\ 1n(2\pi f_h\tau)}{4\pi^2}$$

$$E = \frac{3f_h}{4\pi^2}$$

Annex C

(informative)

Examples and additional variances that have been used to describe frequency instabilities in the time domain

C.1 Introduction

This annex contains basic examples on how to compute the variances used to describe frequency instabilities in the time domain. For more information on this topic and on how to assess the validity of the computations when using larger number of samples, see Riley [B56] and [B57].

C.2 Allan deviation $\sigma_y(\tau)$ examples

Figure C.1 shows a plot of the time deviation between a pair of oscillators as a function of time. The recorded time samples for τ = 1 s are shown in the first column of Table C.1. To compute $\sigma_y(\tau = 1\ \text{s})$, compute the average fractional frequency deviation for x_ks separated by 1 s, then calculate the difference between adjacent $\bar{y}_k$ s and use Equation (A.19) to obtain $\sigma_y(\tau)$. See Table C.1.

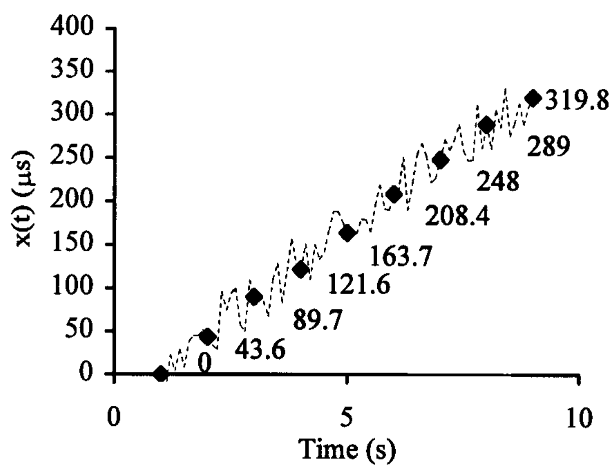

Figure C.1—Plot of *x* (*t*) between a pair of oscillators

Table C.1—Steps to compute $\sigma_y(1\ s)$

x_k (10^{-6})	$\bar{y}_k = (x_{k+1} - x_k)/\tau$ (10^{-6})	$\bar{y}_{k+1} - \bar{y}_k$ (10^{-6})
0	—	—
43.6	$\bar{y}_1 = (x_2 - x_1)/\tau = 43.6$	—
89.7	$\bar{y}_2 = (x_3 - x_2)/\tau = 46.1$	2.5
121.6	$\bar{y}_3 = (x_4 - x_3)/\tau = 31.9$	–14.2
163.7	$\bar{y}_4 = (x_5 - x_4)/\tau = 42.1$	10.2
208.4	$\bar{y}_5 = (x_6 - x_5)/\tau = 44.7$	2.6
248	$\bar{y}_6 = (x_7 - x_6)/\tau = 39.6$	–5.1
289	$\bar{y}_7 = (x_8 - x_7)/\tau = 41.0$	1.4
319.8	$\bar{y}_8 = (x_9 - x_8)/\tau = 30.8$	–10.2

In this example $N = 9$ (number of time samples) and $M = 8$; therefore,

$$\sigma_y(\tau = 1\ s) = \left[\frac{1}{2(7)}\sum_{k=1}^{7}(\bar{y}_{k+1} - \bar{y}_k)^2\right]^{1/2}, \tag{C.1}$$

$$\sigma_y(1\ s) = [3.2 \times 10^{-11}]^{1/2} = 5.67 \times 10^{-6}.$$

For $\tau = 2$ s ($= 2\tau_o$) the procedure is similar: compute the fractional frequency deviation for x_k s separated by 2 s, then calculate the difference between adjacent $\bar{y}_k$ s and use Equation (A.19) to obtain $\sigma_y(\tau)$. See Table C.2.

For this example $M = 4$ since there are a total of four $\bar{y}_k$ s. Therefore,

$$\sigma_y(\tau = 2\ s) = \left[\frac{1}{2(3)}\sum_{k=1}^{3}(\bar{y}_{k+m} - \bar{y}_k)^2\right]^{1/2}, \tag{C.2}$$

$$\sigma_y(2\ s) = [2.12 \times 10^{-11}]^{1/2} = 4.6 \times 10^{-6}.$$

As mentioned in A.3, it is usually more efficient to use overlapped estimates when possible since using overlapped estimates results in a better confidence interval. Figure C.2 illustrates how to compute the $\bar{y}_k$ s for an overlapped estimate of σ_y ($\tau = 2$ s). In this case $m = 2$ ($\tau = 2\tau_o$) and $M = 8$. The $\bar{y}_k$ s and the second difference values are shown in Table C.3.

Table C.2—Steps to compute $\sigma_y(2\ \text{s})$

x_k (10^{-6})	$\bar{y}_k = (x_{k+2} - x_k)/\tau$ (10^{-6})	$\bar{y}_{k+2} - \bar{y}_k$ (10^{-6})
0	—	—
43.6	—	—
89.7	$\bar{y}_1 = (x_3 - x_1)/2 = 44.85$	—
121.6	—	—
163.7	$\bar{y}_3 = (x_5 - x_3)/2 = 37$	–7.85
208.4	—	—
248	$\bar{y}_5 = (x_7 - x_5)/2 = 42.15$	5.15
289	—	—
319.8	$\bar{y}_7 = (x_9 - x_7)/2 = 35.9$	–6.25

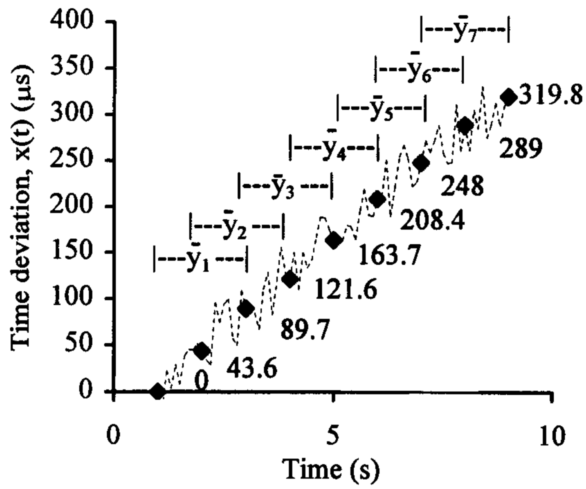

Figure C.2—Computation of $\bar{y}_k$s for overlapped estimates

Table C.3—Steps to compute an overlapped estimate of $\sigma_y(2\ \text{s})$

x_k (10^{-6})	$\bar{y}_k = (x_{k+2} - x_k)/\tau$ (10^{-6})	$\bar{y}_{k+2} - \bar{y}_k$ (10^{-6})
0	—	—
43.6	—	—
89.7	$\bar{y}_1 = (x_3 - x_1)/\tau = 44.9$	—
121.6	$\bar{y}_2 = (x_4 - x_2)/\tau = 39$	—
163.7	$\bar{y}_3 = (x_5 - x_3)/\tau = 37$	–7.85
208.4	$\bar{y}_4 = (x_6 - x_4)/\tau = 43.4$	4.4
248	$\bar{y}_5 = (x_7 - x_5)/\tau = 42.2$	5.15
289	$\bar{y}_6 = (x_8 - x_6)/\tau = 40.3$	–3.1
319.8	$\bar{y}_7 = (x_9 - x_7)/\tau = 35.9$	–6.25

There are a total of ($N - 2m = 5$) second difference values ($\bar{y}_{k+m} - \bar{y}_k$); therefore, the Allan deviation equation becomes

$$\sigma_y(\tau) = \left[\frac{1}{2(N-2m)}\sum_{k=1}^{N-2m}(\bar{y}_{k+m} - \bar{y}_k)^2\right]^{1/2}, \tag{C.3}$$

where $\bar{y}_k = (x_{k+m} - x_k)/\tau$. Equation (C.3) becomes Equation (A.21) when the $\bar{y}_k$ s are expressed in terms of the initial time residual measurements. It is used in this example to help explain the origin of Equation (A.21). Using the values in Table C.3 (last column), Equation (C.3) yields

$$\sigma_y(2\ \text{s}) = \left\{\frac{1}{2(5)}[(-7.85)^2 + 4.4^2 + 5.15^2 + (-3.1)^2 + (-6.25)^2]\right\}^{1/2},$$

$$\sigma_y(2\ \text{s}) = [1.56 \times 10^{-11}]^{1/2} = 3.95 \times 10^{-6}.$$

C.3 Modified Allan deviation Mod $\sigma_y(\tau)$ example

The modified Allan deviation can also be used to characterize frequency stability in the time domain. This deviation uses the average of m adjacent x_k s when computing the stability for $\tau = m\tau_o$. The fractional frequency deviations are then obtained using the $\bar{x}_k$ s. See Figure 3 for computation of $\bar{x}_k$ s and $\bar{y}'_k$ s.

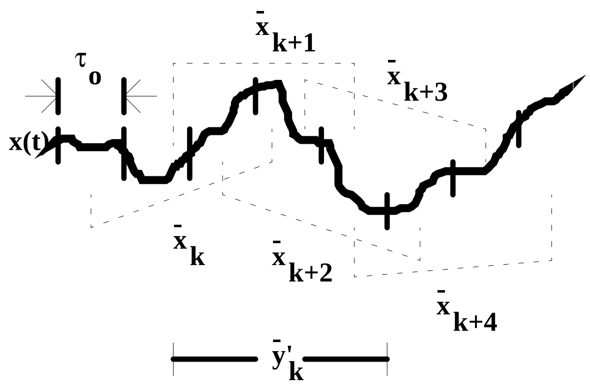

Figure C.3—Method for calculating $\bar{x}_k$s, and $\bar{y}'_k$s for Mod $\sigma_y(\tau)$

Table C.4—Computed $\bar{x}_k$, and $\bar{y}'_k$ values for Mod σ_y(2 s)

x_k (10^{-6})	$\bar{x}_k = (x_{k+1} + x_k)/2$ (10^{-6})	$\bar{y}'_k = (\bar{x}_{k+2} - \bar{x}_k)/\tau$ (10^{-6})	$\bar{y}'_{k+2} - \bar{y}'_k$ (10^{-6})
0	—	—	—
43.6	21.8	—	—
89.7	66.65	—	—
121.6	105.65	$\bar{y}'_1 = (\bar{x}_3 - \bar{x}_1)/2 = 41.93$	—
163.7	142.65	$\bar{y}'_2 = (\bar{x}_4 - \bar{x}_2)/2 = 38$	—
208.4	186.05	$\bar{y}'_3 = (\bar{x}_5 - \bar{x}_3)/2 = 40.2$	–1.73
248	228.2	$\bar{y}'_4 = (\bar{x}_6 - \bar{x}_4)/2 = 42.78$	4.78
289	268.5	$\bar{y}'_5 = (\bar{x}_7 - \bar{x}_5)/2 = 41.23$	1.03
319.8	304.4	$\bar{y}'_6 = (\bar{x}_8 - \bar{x}_6)/2 = 38.1$	–4.68

Table C.4 shows the computed $\bar{x}_k$s, and $\bar{y}'_k$s for $\tau = 2$ s. The modified Allan deviation can then be obtained by using Equation (A.19) and the fact that $m = 2$ and the equivalent M is $N - 3m + 1$:

$$\text{Mod } \sigma_y(\tau) = \left[\frac{1}{2(N-3m+1)} \sum_{k=1}^{N-3m+1} (\bar{y}'_{k+m} - \bar{y}'_k)^2\right]^{1/2} \tag{C.4}$$

$$\text{Mod } \sigma_y(2\text{ s}) = \left\{\frac{1}{2(4)}[(-1.73)^2 + 4.78^2 + 1.03^2 + (-4.68)^2]\right\}^{1/2}$$

$$\text{Mod } \sigma_y(2\text{ s}) = [6.1 \times 10^{-12}]^{1/2} = 2.47 \times 10^{-6}.$$

Equation (C.4) becomes Equation (A.22) when expressing the $\bar{y}'_k$s in terms of the initial time residual measurements. It is used in this example to help explain the origin of Equation (A.22).

C.4 Total deviation $\hat{\sigma}_{y,\,\mathrm{TOTAL}}(\tau)$ example

In Annex A it was recommended that the total deviation $\hat{\sigma}_{y,\,\mathrm{TOTAL}}(\tau)$ be used to characterize fractional frequency fluctuations when τ exceeds 10% of the data sample. $\hat{\sigma}_{y,\,\mathrm{TOTAL}}(\tau)$ extends the x_k sequence at both ends by reflection about the endpoints to provide a better estimate of frequency stability.

As an example, compute $\hat{\sigma}_{y,\,\mathrm{TOTAL}}(2\ \mathrm{s})$ for $x(t)$ in Figure (C.4). This data set is different from the one used in the previous examples. In this case x_1, x_3, and x_5 almost fall into a line; therefore, the value for $\sigma_y(\tau)$ will be negatively biased (too optimistic).

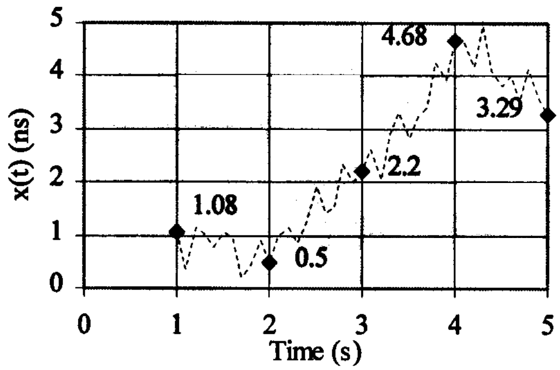

Figure C.4—*x*(*t*) as a function of time

For this data set, $N = 5$ and $m_{max} = (5 - 1)/2 = 2$. There are only two extra x'_k to adjoin, namely, $x'_0 = 2x_1 - x_2 = 1.66$ ns, $x'_6 = 2x_5 - x_4 = 1.90$ ns. According to Equation (A.25),

$$\hat{\sigma}_{y,\,\mathrm{TOTAL}}(2\ \mathrm{s}) = \left\{\frac{1}{2(3)}[(\bar{y}'_2 - \bar{y}'_0)^2 + (\bar{y}'_3 - \bar{y}'_1)^2 + (\bar{y}'_4 - \bar{y}'_2)^2]\right\}^{1/2}$$

where

$$\bar{y}'_0 = \frac{(x'_2 - x'_0)}{2} = \frac{(0.50 - 1.66)}{2} = -0.58 \times 10^{-9},$$

$$\bar{y}'_1 = \frac{(x'_3 - x'_1)}{2} = \frac{(2.20 - 1.08)}{2} = 0.56 \times 10^{-9},$$

$$\bar{y}'_2 = \frac{(x'_4 - x'_2)}{2} = \frac{(4.68 - 0.50)}{2} = 2.09 \times 10^{-9},$$

$$\bar{y}'_3 = \frac{(x'_5 - x'_3)}{2} = \frac{(3.29 - 2.20)}{2} = 0.545 \times 10^{-9},$$

$$\bar{y}'_4 = \frac{(x'_6 - x'_4)}{2} = \frac{(1.90 - 4.68)}{2} = -1.39 \times 10^{-9}.$$

Therefore,

$$\hat{\sigma}_{y,\,\mathrm{TOTAL}}(2\ \mathrm{s}) = \left\{\frac{1}{6}[(2.67)^2 + (0.015)^2 + (3.48)^2]\right\}^{1/2} \times 10^{-9} = 1.79 \times 10^{-9}.$$

This value can be compared to the value obtained for the Allan deviation:

$$\sigma_y(2\ \text{s}) = \left(\frac{1}{2}[\bar{y}_3 - \bar{y}_1]^2\right)^{1/2} = \frac{1}{\sqrt{2}}(0.545 - 0.56) \times 10^{-9},$$

$$\sigma_y(2\ \text{s}) = 1.06 \times 10^{-11}.$$

(Note that $\bar{y}_1 = \bar{y}'_1$, $\bar{y}_3 = \bar{y}'_3$.) $\sigma_y(2\ \text{s})$ is seriously negatively biased by two orders of magnitude compared to $\hat{\sigma}_{y,\ \text{TOTAL}}(2\ \text{s})$.

A slight negative bias in $\hat{\sigma}_{y,\ \text{TOTAL}}(\tau)$ has been found for flicker frequency modulation noise and random walk frequency modulation noise. It is possible to remove this bias if the noise type is assumed to be known (see Howe and Greenhall [B37]).

C.5 Other variances

Several other variances have been introduced by workers in this field. In particular, before the introduction of the two-sample variance, it was standard practice to use the sample variance s^2, defined as

$$s^2 = \int_0^{f_h} S_y(f)\left(\frac{\sin \pi f \tau}{\pi f \tau}\right)^2 df. \tag{C.5}$$

In practice it may be obtained from a set of measurements of the frequency of the oscillator as

$$s^2 = \frac{1}{N}\sum_{i=1}^{N}(y_i - \bar{y})^2. \tag{C.6}$$

The sample variance diverges for some types of noise and, therefore, is not generally useful.

Other variances based on the structure function approach can also be defined (see Lindsey and Chie [B51]). For example, there are the Hadamard variance, the three-sample variance, and the high-pass variance (see Rutman [B62]). They are occasionally used in research and scientific works for specific purposes, such as differentiating between different types of noise and for dealing with systematics and sidebands in the spectrum.

Annex D

(informative)

Confidence limits of measurements

A simple method to compute the confidence interval for $\sigma_y(\tau)$ (see Lesage and Audoin [B45]), which assumes a symmetric (Gaussian) distribution, uses the relation

$$I_\alpha \cong \sigma_y(\tau)\kappa_\alpha M^{-1/2}, \tag{D.1}$$

where

I_α is the confidence interval,
κ is a constant,
α is an integer that depends on the type of noise (see Annex B),
M is the total number of data points used in the estimate.

For a 1σ or 68% confidence interval, the values for κ_α are

$\kappa_2 = 0.99$,

$\kappa_1 = 0.99$,

$\kappa_0 = 0.87$,

$\kappa_{-1} = 0.77$,

$\kappa_{-2} = 0.75$.

As an example of the Gaussian model with $M = 100$, $\alpha = -1$ (flicker frequency noise), and $\sigma_y(\tau = 1\text{ s}) = 1 \times 10^{-12}$, one may write

$$I_\alpha \cong \sigma_y(\tau)(0.77)(100)^{-1/2} = \sigma_y(\tau)\ (0.077),$$

which gives

$$\sigma_y(\tau = 1\text{ s}) = (1 \pm 0.08)\ 10^{-12}.$$

This analysis for $\sigma_y(\tau)$ is valid only for $M \geq 10$. If M is small, then the plus and minus confidence intervals become sufficiently asymmetric, and the κ_α coefficients are not valid. However, these confidence intervals can be calculated (see Lesage and Audoin [B45]).

Another way of computing confidence intervals for $\sigma_y(\tau)$ is to use the chi-squared distribution. The estimated Allan variance has a chi-squared distribution function given by Equation D.2. The number of degrees of freedom for a specific noise process and number of samples can be computed and then used in Equation D.2 to compute the confidence interval (see Howe [B34]):

$$\chi^2 = (\mathrm{df})\frac{\hat{\sigma}_y^2}{\sigma_y^2}, \tag{D.2}$$

where

df is the number of degrees of freedom,

$\hat{\sigma}_y^2$ is the estimated (measured) Allan variance,

σ_y^2 is the true Allan variance.

Table D.1 shows empirical equations to compute the number of degrees of freedom for different types of noise processes (see Howe [B34]). This table is valid only for overlapped estimates of the Allan variance.

To compute the confidence interval for $\hat{\sigma}_y(\tau = 1\text{ s}) = 10^{-12}$, for flicker frequency noise, $N = 101$, and $\tau_o = 0.5$ s ($m = 2$), first find the number of degrees of freedom using Table D.1:

$$\mathrm{df} = \frac{5N^2}{4m(N+3m)} \tag{D.3}$$

$$\mathrm{df} = \frac{5(101^2)}{4(2)(101+3(2))} = 59.6\,.$$

Table D.1—Empirical equations for the number of degrees of freedom of the Allan variance estimate (see Howe [B34])

Noise process	Degrees of freedom[a]
White phase modulation	$\frac{(N+1)(N-2m)}{2(N-m)}$
Flicker phase modulation	$\exp\left[\ln\left(\frac{N-1}{2m}\right)\ln\left(\frac{(2m+1)(N-1)}{4}\right)\right]^{1/2}$
White frequency modulation	$\left[\frac{3(N-1)}{2m} - \frac{2(N-2)}{N}\right]\frac{4m^2}{4m^2+5}$
Flicker frequency modulation	$\frac{2(N-2)^2}{2.3N-4.9}$ for m = 1 $\frac{5N^2}{4m(N+3m)}$ for $m \geq 2$
Random walk frequency modulation	$\frac{N-2}{m}\frac{(N-1)^2 - 3m(N-1) + 4m^2}{(N-3)^2}$

[a]N = number of samples, and $m = \tau/\tau_o$.

For a 1σ (68%) confidence interval, the χ^2 values needed are $\chi^2(0.16)$ and $\chi^2(1–0.16)$. These values can be obtained from numerical tables of the chi-squared distribution function or from various computer programs. For this example, $\chi^2(0.16) = 48.25$, and $\chi^2(0.84) = 69.73$.

Therefore,

$$\chi^2(0.16) < \frac{59.6\hat{\sigma}_y^2}{\sigma_y^2} < \chi^2(0.84)\,, \tag{D.4}$$

or

$$\frac{59.6\hat{\sigma}_y^2}{\chi^2(0.84)} < \sigma_y^2 < \frac{59.6\hat{\sigma}_y^2}{\chi^2(0.16)}\,, \tag{D.5}$$

$$0.85\hat{\sigma}_y^2 < \sigma_y^2 < 1.24\hat{\sigma}_y^2\,. \tag{D.6}$$

Other methods have been developed for computing the confidence interval of Mod $\sigma_y(\tau)$ (see Walter [B76] and Greenhall [B31]). Table D.2 shows a comparison of confidence intervals for $\sigma_y(\tau)$ (no overlap and full overlap) and Mod $\sigma_y(\tau)$ for white phase modulation, flicker phase modulation, and white frequency modulation noise processes (see Lesage and Audoin [B45], Howe [B34], Stein [B64], Walter [B76], and Weiss [B77]). As shown in Table D.2, overlapped estimates improve the confidence intervals for specific values of M and m. Although $\sigma_y(\tau)$ usually provides a better estimate (confidence interval) than Mod $\sigma_y(\tau)$ for fractional fluctuations (see Table D.2 for white phase modulation and flicker phase modulation), the confidence interval for absolute fluctuations is approximately similar. The reason is that the value of Mod $\sigma_y(\tau)$ is typically much smaller than $\sigma_y(\tau)$ for white and flicker phase modulation noise processes.

Table D.2—Confidence intervals for $\sigma_y(\tau)$ (no overlap and full overlap) and Mod $\sigma_y(\tau)$ (see Lesage and Audoin [B45], Howe [B34], Stein [B64], Walter [B76], and Weiss [B77])

***n* = 1025**	**No overlap ± for 68% $\sigma_y(\tau)$ White phase modulation**	**Full overlap – for 68% $\sigma_y(\tau)$ White phase modulation**	**Full overlap + for 68% $\sigma_y(\tau)$ White phase modulation**	**Full overlap – for 68% Mod $\sigma_y(\tau)$ White phase modulation**	**Full overlap + for 68% Mod $\sigma_y(\tau)$ White phase modulation**
m = 2	4.4%	2.9%	3.2%	3.1%	3.4%
m = 8	8.7%	2.9%	3.2%	5.2%	6.1%
m = 32	17.4%	3.0%	3.4%	9.7%	14%
m = 128	34.9%	3.1%	3.6%	18%	41%
***n* = 1025**	**Flicker phase modulation**	**Flicker phase modulation**	**Flicker phase modulation**	**Flicker phase modulation**	**Flicker phase modulation**
m = 2	4.4%	2.9%	3.1%	3.0%	3.3%
m = 8	8.7%	3.6%	4.0%	5.7%	6.8%
m = 32	17.4%	5.2%	6.1%	11%	16%
m = 128	34.9%	8.4%	11%	20%	50%
***n* = 1025**	**White frequency modulation**	**White frequency modulation**	**White frequency modulation**	**White frequency modulation**	**White frequency modulation**
m = 2	3.8%	2.8%	3.0%	3.0%	3.2%
m = 8	7.7%	4.8%	5.6%	5.8%	7.0%
m = 32	15.3%	8.8%	12%	11%	16%
m = 128	30.6%	16%	32%	20%	51%

The confidence limits for frequency domain measures (spectral densities) can be approximated by

$$1 \pm \frac{k}{\sqrt{N\beta}}, \tag{D.7}$$

where

N is the number of averages,
$\beta = 1$ for FFT spectrum analyzers and β = (resolution bandwidth)/(video bandwidth) for swept spectrum analyzers,
$k = 1$ for 1σ or 68% confidence and $k = 2$ for 2σ or 95% confidence (see Walls [B75]).

Annex E

(informative)

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