Characterization of Frequency Stability In Precision Frequency Sources

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Invited Paper

This paper presents a short review of the progress that has occurred over the past 25–30 years in both the theoretical and practical characterization of frequency stability of precision frequency sources. The emphasis is on the evolution of ideas and concepts for the characterization of random noise processes in such standards in the time domain and the Fourier frequency domain, rather than a rigorous mathematical treatment of the problem. Numerous references to the mathematical treatments are made.

I. Introduction

High precision frequency standards have undergone tremendous advances during the decades since the advent of the first laboratory cesium beam clock in 1955. Thousands of atomic clocks, such as the cesium beam and the optically pumped rubidium standards manufactured by industry, are in routine use today. The ultrastable hydrogen maser is also used on a large scale for very demanding applications. Quality quartz-crystal-controlled oscillators have also shown such progress in stability that they can sometimes compete with rubidium clocks. These devices are used in applications such as: fundamental metrology, telecommunications systems, space missions, radars, broadcasting, etc.

By the early-1960's it was clearly recognized that there was a real need for a common set of frequency stability characterization parameters and for related measurement test-sets [1], [2].

These parameters were needed for at least two main purposes:

1) to allow for meaningful comparisons between similar devices developed by different laboratories, or between different devices in a given application;
2) to access application performance in terms of the measured oscillator frequency stability.

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Related experimental test sets with well-defined characteristics were of course needed in order to unambiguously measure the various frequency stability parameters.

This paper presents a short review of the progress that has occurred during the last 25–30 years both in terms of theoretical characterization of frequency stability and of experimental measurement test sets.

II. Charcterization of Frequency Stability

In this paper no attempt will be made to give mathematical developments that can be found in many references [1]–[47]. Reference [21] contains many of the original papers with errata sheets. It attempts to point out inconsistencies between the notation of these papers and the updated recommendations of IEEE [7] and CCIR [38]. We will concentrate mainly on the evolution of ideas and concepts and try to highlight the key milestones with a minimum of mathematical symbols and equations.

In simple terms, the practical problem is how to characterize the properties of the output signal from a real oscillator. The output signal from an ideal noise-free non-drifting oscillator would be a pure sine wave, but any real device, even the most stable, is disturbed by unavoidable processes such as random noises, drifts due to aging and/or environmental effects. This paper will be mainly devoted to the characterization of frequency instabilities due to random noises which exist in all kinds of devices. Hence, the first step is to develop a tractable mathematical model for the quasi-sinusoidal output signal of an oscillator.

III. Output Signal Model

A relatively simple model that was introduced in the early 1960's and has found wide acceptance is

$$V(t) = (V_0 + \varepsilon(t))\sin[2\pi\nu_0 + \phi(t)] \tag{1}$$

where $\phi(t)$ is a random process denoting phase noise [6], [7], [21], [33], [37], [38], [43], V_0 and ν_0 are the nominal amplitude and frequency respectively; and amplitude noise characterized by $\varepsilon(t)$ that can usually be neglected in high performance sources. (In this treatment we assume that frequency drift, if any, has been removed.) Such a quasi-sinusoidal signal has an instantaneous frequency defined as

$$\nu(t) = \frac{1}{2\pi}\frac{d}{dt}(2\pi\nu_0 t + \phi(t)) = \nu_0 + \frac{1}{2\pi}\frac{d\phi(t)}{dt}. \quad (2)$$

Frequency noise is the random process defined by

$$\Delta\nu(t) \equiv \nu(t) - \nu_0 = \frac{1}{2\pi}\frac{d\phi(t)}{dt} \quad (3)$$

which exists simultaneously with and has properties similar to phase noise, as will be seen later. Very often it is useful to introduce the normalized dimensionless frequency fluctuations,

$$y(t) = \frac{\Delta\nu(t)}{\nu_0}. \quad (4)$$

This quantity remains unchanged under frequency multiplication or division and can be used as a basis for comparisons of oscillators at different nominal frequencies.

Since we have modeled phase and frequency fluctuations by random processes, we are now in a position to use the various statistical tools which allow fluctuation characterization, such as correlation functions, spectral densities, averages, standard deviations and variances etc. Many textbooks exist on this subject. Now the problem of frequency (or phase) instability characterization is to introduce meaningful and practical (i.e. measurable) parameters for describing the statistical properties of $\phi(t), x(t), \Delta\nu(t)$, or $y(t)$.

IV. The Great Dichotomy

Users of frequency standards in various fields recognized early on that they needed two kinds of parameters in order to meet requirements of different applications—namely spectral parameters (related to the spread of signal energy in the Fourier frequency spectrum) and time parameters (allowing assessment of the stability over a given time interval).

Therefore two sets of parameters have been introduced as tools for oscillator characterization:

1) spectral densities of phase and frequency fluctuations, in the so-called Fourier frequency domain.
2) variances (or standard-deviation) of the averaged frequency fluctuations in the time domain.

We first view briefly these two different kinds of parameters and then describe the mathematical relationships between them together with the related experimental consequences. A key point is the integral relationship which allows us to derive the variances from the knowledge of the spectral densities.

V. Fourier Frequency Domain

In the Fourier frequency domain, phase and frequency fluctuations can be characterized by the respective one-sided spectral densities, $S_\phi(f)$ and $S_{\Delta\nu}(f)$, which are related by the simple law [6], [7], [21], [33], [37], [38],

$$S_{\Delta\nu}(f) = f^2 S_\phi(f) \quad (5)$$

which corresponds to the time derivative between $\phi(t)$ and $\Delta\nu(t)$. The spectral density $S_y(f)$ is also widely used and is very simply related to $S_{\Delta\nu}(f)$ and $S_\phi(f)$ by

$$S_y(f) = \frac{1}{\nu_0^2}S_{\Delta\nu}(f) = \frac{f^2}{\nu_0^2}S_\phi(f). \quad (6)$$

(Note the word "frequency" is used with two different meanings which should not be confused. $\nu(t)$ is the time-dependent instantaneous frequency of the oscillator, and f is the time-independent Fourier frequency that appears in any spectral density. The spectral density is the Fourier transform of the autocorrelation function [6], [33].)

It has been shown from both theoretical considerations and experimental measurements, that the spectral densities due to random noise of all high stability frequency standards can be modelled by the power law model where the spectral densities vary as a power of f. More specifically, $S_y(f)$ can be written as the sum:

$$S_y(f) = \sum_{\alpha=-2}^{+2} h_\alpha f^\alpha \quad (7)$$

for $0 \leq f \leq f_h$ where f_h is an upper cutoff frequency. For a given type of oscillator two or three terms of the sum are usually dominant. Each term is related to a given noise source in the oscillator (internal and/or external white noise, flicker noise,...). The most common noise types encountered in practical sources are given in Table 1. Of course, power laws can sometimes lead to "mathematical pathologies" (divergence of integrals) when they are integrated from $f = 0$ to $f = \infty$, but this is only a limitation of the model that can be overcome by physical considerations (limited bandwidth and duration, for example).

$S_y(f)$ was proposed in 1971 by the IEEE as the recommended frequency stability measure in the Fourier frequency domain [6]. The updated version given in [7] is in general agreement with the recommendations of CCIR [37], [38].

For stationary Gaussian random processes, the spectral density (or the autocorrelation function) contains the maximum information about the process. The variances that will be defined later are all related to the spectral densities via integrals and transfer functions. Some information is, however, lost in the process.

Spectral densities of phase or frequency are measured by a spectrum analyzer (analog or fast Fourier transform) following some kind of demodulation of $\phi(f)$ or $\Delta\nu(t)$. Numerous experimental tests sets have been developed for that purpose [8], [18], [23], [39]. Figure 1 shows $S_y(f)$ for the five common power-law noise processes listed in Table 1. Specific techniques for measuring $S_\phi(f)$ (or equivalently $S_y(f)$) are described in [8], [9], [13], [18], [19], [21]–[23], [43]–[45]. Particular attention is focused on describing the errors in such measurements and in

Table 1 Listing of the five common types of noise found in practical sources. The asymptotic forms of $\sigma_y^2(\tau)$ for various power-law noise types and two filter types are also listed. Note: $\omega_h/2\pi = f_h$ is the measurement system bandwidth—often called the high-frequency cutoff. ln $\equiv$ log

			$\sigma_y^2(r)$			
			$\omega_h r >> 1$	$\omega_h r >> 1$	$\omega_h r << 1$	$\omega_h r << 1$
Name of Noise	α	$S_y(f)$	Infinite Sharp Filter	Single Pole Filter	Infinite Sharp Filter	Single Pole Filter
White phase	2	$h_2 f^2$	$\frac{3 f_h h_2}{(2\pi)^2 \tau^2}$	$\frac{3 f_h h_2}{(2\pi)^2 \tau^2}$	$2/5\pi^2 f_h^5 \tau^2 h_2$	$\frac{f_h^2 h_2}{2\tau}$
Flicker phase	1	$h_1 f$	$\frac{(1.038+3\ln(\omega_h \tau))h_1}{(2\pi)^2 \tau^2}$	$\frac{(3\ln(\omega_h \tau))h_1}{(2\pi)^2 \tau^2}$	$1/2\pi^2 f_h{}^4 \tau^2 h_1$	$2 f_h^2 (\ln(2)) h_1$
White frequency	0	h_o	$\frac{h_o}{2\tau}$	$\frac{h_o}{2\tau}$	$2/3\pi^2 f_h{}^3 \tau^2 h_0$	$2/3\pi^2 f_h{}^2 \tau h_0$
Flicker frequency	-1	$h_{-1} f^{-1}$	$2(\ln(2))h_{-1}$	$2(\ln(2))h_{-1}$	$\pi^2 f_h{}^2 \tau^2 h_{-1}$	$8\pi^2 f_h{}^2 \tau^2 h_{-1}$
Random-walk frequency	-2	$h_{-2} f^{-2}$	$\frac{2\pi^2 \tau h_{-2}}{3}$	$\frac{2\pi^2 \tau h_{-2}}{3}$	$2\pi^2 f_h \tau^2 h_{-2}$	$2\pi^2 f_h \tau^2 h_{-2}$

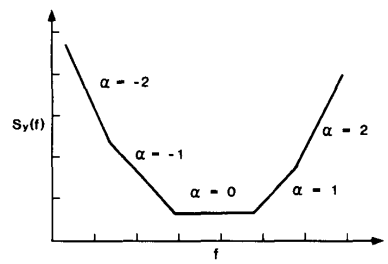

Fig. 1. Spectral density of frequency fluctuations $S_y(f)$ for the five common noise types.

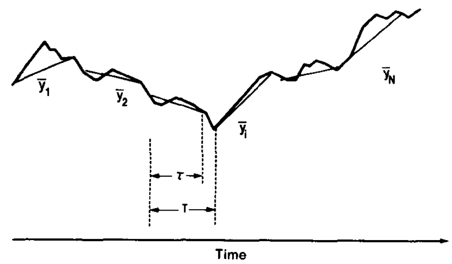

Fig. 2. Measurements process for the computation of sample variances. The phase difference, between two oscillators, $x(t)$, is plotted as a function of time. Here T is the total cycle time and τ is the averaging time for each measurement. The fractional frequency difference y_k is given by $\overline{y}_k = \frac{x(t_k+\tau)-x(t_k)}{\tau}$.

determining $S_y(f)$ or $S_\phi(f)$ for a single oscillator in [8], [18], [23]. Confidence intervals for spectral density measurement of nonwhite noise are treated in [41].

VI. Time Domain

Time-domain characterization of frequency stability is widely used since it answers the obvious question: what is the stability over a time interval τ for a given application? (τ can range from milliseconds to months and years according to the application.)

In the time domain, the basic measurement apparatus is a digital counter that yields results that can be related to $\overline{y}_i$, the ith average value of $y(t)$ over a time interval τ beginning at time t_i (any physical measurement has a finite duration τ that cannot approach zero; instantaneous frequency cannot be measured). Figure 2 shows the basic measurement cycle. Simple counting techniques are, however, severely limited in precision. Heterodyne techniques offer much higher accuracy at the expense of increased complexity [9], [20], [21], [24], [42].

In order to assess frequency stability over a time interval τ (the sample time), it is necessary to make a series of measurements, each of duration τ, which yields the results $\overline{y}_i$ with $i = 1-N$. Of course, due to the random fluctuations of $y(t)$, the $\overline{y}_i$'s are samples of a random variable and frequency stability over τ which can only be defined from a measure of the dispersion. A widely used statistical tool for that is the variance, σ^2, or the square root of the variance, σ, normally called the standard deviation.

A very specific problem for oscillator characterization is that several kinds of variances have been introduced by several authors since the early 1960's and thus it is necessary to give a clear picture of the relationships (if any) between the variances and the spectral densities.

A. True Variance

The true variance is a theoretical parameter denoted as $I^2(\tau)$ and simply defined as: $I^2(\tau) = <\overline{y}_k^2>$. When $y(t)$ has a zero mean, the bracket < > denotes an infinite time average made over one sample of $y(t)$. For stationary frequency fluctuations around $\nu_0, I^2(\tau)$ decreases from $<y^2(t)>$ for $\tau = 0$ to $I^2(\tau) = 0$, for $\tau \to \infty$ where fluctuations are completely averaged away as shown in curve a of Fig. 3. However, despite its mathematical simplicity, the true variance is not really useful for experimental purposes since it approaches infinity for all real oscillators as shown by curve b in Fig. 3. Practical estimators of the time-domain stability relying on the sample variance concept were introduced in 1966 to avoid the divergence of the true variance observed in most sources [5].

B. Sample Variance

The sample variance is a more practical estimate of time-domain stability based upon a finite number of N samples $\overline{y}_k (k = 1-N)$ than the time variance. Each sample has a duration τ, and the kth sample begins at t_k; the $(k+1)$th

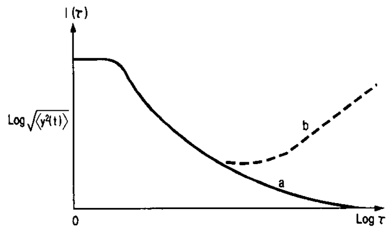

Fig. 3. Frequency instability in the time-domain. (a) Square root of the true variance for *stationary* frequency noise. (b) Performance of practical frequency sources, i.e., reality.

sample begins at $t_{k+1} = t_k + T$; the dead time between two successive samples is then $T - \tau$. The quantity T is the repetition interval for individual measurements of duration τ as shown in Fig. 2. The sample variance is defined as

$$\sigma_y^2(N,T,\tau) = \frac{1}{N-1}\sum_{i=1}^{N}(\overline{y}_i - \frac{1}{N}\sum_{j=1}^{N}\overline{y}_j)^2. \quad (8)$$

This quantity is itself a random variable, N being the sample size. Its average can be used as a measure of frequency stability over a time interval τ:$<\sigma_y^2(N,T,\tau)>$. (See [3] for more detailed considerations about the various possible definitions of sample variances and their respective advantages or limitations—biased or unbiased estimates.)

C. Allan Variance

With the sample variance as defined above, we are faced with several possible values for the parameters N and T (besides τ, which is the duration over which stability is measured). To achieve the goal of recommending a unique time-domain measure that can be used unambiguously in laboratories all around the world, some choices must be made.

Following the pioneering work of David Allan in 1966 [5], the IEEE subcommittee on frequency stability recommended in 1971 to use the average of the variance with $N = 2$ and adjacent samples (that is $T = \tau$, or zero dead time) [6]. The resulting measure is denoted as

$$\sigma_y^2(\tau) = \frac{1}{2} <(\overline{y}_2 - \overline{y}_1)^2> \quad (9)$$

and known as the Allan variance, or two-sample variance, since pairs of adjacent measurements are grouped together.

$\sigma_y^2(\tau)$ is also a theoretical measure since infinite duration is implied in the average denoted as $< >$. However, it has a much greater practical utility than $I^2(\tau)$ since it exists for all the spectral density power laws encountered in real oscillators (Table 1) including flicker frequency noise. This will be shown later from the mathematical relationship between frequency- and time-domain parameters. Moreover, simple experimental estimates may be derived for $\sigma_y^2(\tau)$ since groups of only two measurements are involved. The choice of $N = 2$ in the preferred stability time domain measure is really the key feature in the definition of $\sigma_y^2(\tau)$. Although there are no recommended values for the measurement bandwidth, f_h, it has to be specified with any experimental results for comparison purposes (and also because the result can be f_h-dependent for some kinds of noise.)

D. Estimates of the Allan Variance

Experimentally only estimates of $\sigma_y^2(\tau)$ can be obtained from a finite number of samples $\overline{y}_k$ taken over a finite duration. Therefore an inherent statistical uncertainty (error bars) exists when m values of $\overline{y}_k$ are used to estimate $\sigma_y^2(\tau)$. A widely used estimator is:

$$\sigma_y^2(\tau, m) = \frac{1}{2(m-1)}\sum_{i=1}^{m-1}\left(\overline{y}_{i+1} - \overline{y}_i\right)^2. \quad (10)$$

This quantity is itself a random variable whose variance (the variance of the variance) may be used to calculate the error bars on the plot of $\sigma_y^2(\tau)$ versus τ. This subject was treated in great detail by Lesage and Audoin in 1973, when they calculated the error bars for Gaussian noises characterized by power law spectral densities [11]. An additional treatment of the confidence limits is to be found in [21]. For long-term stability (τ of the order of days, months, or even a year) the size of m is severely limited. In any case, m should be stated with any results to avoid ambiguity and to allow for meaningful comparisons.

Estimates of the bias in experimental measurements made with dead time have been made by Barnes [12], as shown below:

$$\sigma_y^2(\tau) = \frac{<\sigma_y^2(2,T,\tau)>}{B_2}. \quad (11)$$

B_2 is the bias function given in [12]. This is of practical interest since counting techniques usually have nonzero dead time between successive measurements. Estimates of the biases caused by unequally spaced data are given by Barnes and Allan [25]. Most precision measurement techniques eliminate the problems of dead time and hence do not require these bias functions [20], [24], [42]. Techniques for determining $\sigma_y(\tau)$ for an individual oscillator are treated in [17], [18], [21], [23], [29], [44] as well as many of the other references. Figure 4 shows the dependence of $\sigma_y(\tau)$ on measurement time for the five common power-low noise process in the limit that $2\pi f_h \tau$ is large compared to 1.

E. Comments on $\sigma_y(\tau)$

The slope of $\sigma_y(\tau)$ versus τ is virtually the same for $\alpha = 1$ and $\alpha = 2$. As a consequence $\sigma_y(\tau)$ is not useful for distinguishing between these noise types. With both noise types, frequency (or phase) fluctuations at $f = f_h$ dominate $\sigma_y(\tau)$, even for extremely long measurement times. Changes in the average frequency over long times do not bias the characterization of short-term frequency stability as occurs with $I^2(\tau)$. The determination of $\sigma_y(\tau)$ is dependent on the noise bandwidth and, in the limit $2\pi f_h \tau \leq 1$, the type of low-pass filter. This is illustrated for $\alpha = 2$ in Fig. 5. $\sigma_y^2(\tau)$ is a very efficient estimator for noise types $\alpha = 0, -1, -2$ but diverges for $\alpha \leq -3$.

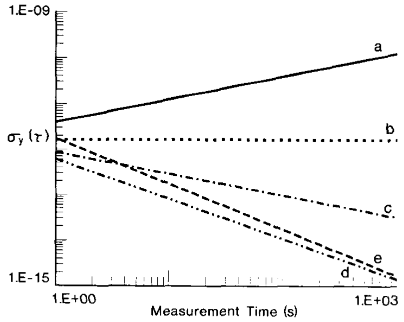

Fig. 4. $\sigma_y^2(\tau)$ versus τ for the five common power-law noise types in the limit that $2\pi f_h \tau$ is large compared to 1 and an infinitely sharp filter is used. Curve a is for random-walk frequency modulation, $S_y(f) = h_{-2}f^{-2}$. Curve b is for flicker frequency modulation, $S_y(f) = h_{-1}f^{-1}$. Curve c is for white frequency modulation, $S_y(f) = h_0$. Curve d is for flicker phase modulation, $S_y(f) = h_1 f$ and a bandwidth of 16 Hz. Curve e is for white phase modulation, $S_y(f) = h_2 f^2$ and a bandwidth of 16 Hz. $h_{-2} = h_{-1} = h_0 = h_1 = h_2 = 2 \times 10^{-24}$.

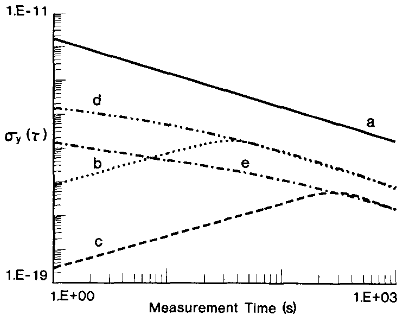

Fig. 5. $\sigma_y(\tau)$ for white phase modulation ($\alpha = 2$) as a function of measurement time, τ, and measurement bandwidth, f_h. Curves a, b, and c have an infinitely sharp filter with width, $f_h = 16$ Hz, $f_h = 0.016$ Hz, $f_h = 0.0016$ Hz, respectively. Curves d and e have a single pole filter width, $f_h = 0.016$ Hz and $f_h = 0.0016$, respectively. $h_2 = 2 \times 10^{-24}$

More convergent variances are introduced in F below and in Section VIII.

F. Modified Allan Variance

The relatively poor discrimination of $\sigma_y(\tau)$ against white and flicker phase noise prompted the development of the modified Allan variance, mod $\sigma_y^2(\tau)$, as shown below, in 1981 [30], [31].

$$\begin{aligned}
&\text{mod } \sigma_y^2(n\tau_0) \\
&\quad = \frac{1}{2n^2\tau_0^2}\left\langle \left[\frac{1}{n}\sum_{i=1}^{n}(x_{i+2n} - 2x_{i+n} + x_i)\right]^2 \right\rangle \\
&\quad = \frac{1}{2}\left\langle \frac{1}{n^2}(y_{n+1}(\tau) - y_1(\tau) + y_{n+2}(\tau) \right. \\
&\quad \left. - y_2(\tau) \cdots y_{2n}(\tau) - y_n(\tau)) \right\rangle \qquad (12)
\end{aligned}$$

Here the x_i's are the time variations measured at intervals t_0 and $\bar{t}_i(\tau) = (\bar{x}_{i+n\tau_0} - \bar{x}_i)/(n\tau_0)$. See Fig. 6. $\bar{x}_i$ is the phase averaged over n adjacent measurements of duration τ_0. Thus mod $\sigma_y^2(\tau)$ is proportional to the second difference of the phase averaged over a time $n\tau_0$. Viewed from the frequency domain, mod $\sigma_y^2(\tau)$ is proportional to the first difference of the frequency averaged over n adjacent samples. If $n = 1$ then $\sigma_y(\tau)$ is equal to mod $\sigma_y^2(\tau)$. This measurement process results in an equivalent noise bandwidth of f_h/n when $2\pi f_n \tau_0 >> 1$. Figure 7 shows the dependence of mod $\sigma_y(n\tau_0)$ on measurement time $n\tau_0$ for the five common noise types, in the limit that $2\pi f_h \tau_0 >> 1$ [26]–[28], [31], [34]. Figure 8 shows the ratio of mod $\sigma_y^2(n\tau_0)/\sigma_y^2(n\tau_0)$ versus n.

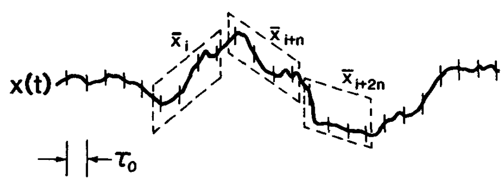

Fig. 6. Measurement process for determining x_i's used in the determination of mod $\sigma_y(\tau)$.

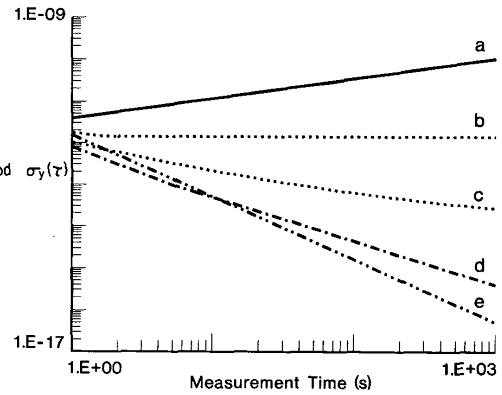

Fig. 7. Mod $\sigma_y(\tau)$ versus τ for the five common power-law noise types in the limit that $2\pi f_h \tau_0$ is large compared to 1 and an infinitely sharp filter is used. Curve a is for random walk frequency modulation, $S_y(f) = h_{-2}f^{-2}$. Curve b is for flicker frequency modulation, $S_y(f) = h_{-1}f^{-1}$. Curve c is for white frequency modulation, $S_y(f) = h_0$. Curve d is for flicker phase modulation, $S_y(f) = h_1 f$ and a bandwidth of 16 Hz. Curve e is for white phase modulation, $S_y(f) = h_2 f^2$ and a bandwidth of 16 Hz. $h_{-2} = h_{-1} = h_0 = h_1 = h_2 = 2\times 10^{-24}$. See Fig. 4 for the values of $\sigma_y(\tau)$ for the same noise levels.

G. Comments on mod $\sigma_y(\tau)$

Mod $\sigma_y(n\tau_0)$ behaves very similarly to $\sigma_y(n\tau_0)$ for $\alpha = 0, -1, -2$ and -3. Noise types $\alpha = 1$ and $\alpha = 2$ are easily separated using mod $\sigma_y(n\tau_0)$. this approach is actually a software realization of the variable noise bandwidth proposal of [32]. In the presence of noise types $\alpha \geq 1$, mod $\sigma(n\tau_0)$ depends on τ_0. By way of illustration, assume $\alpha = 2$; then mod $\sigma_y(10n_i, \tau_0) = 1/10^{-1/2}$ mod $\sigma_y(n_i, 10\tau_0)$. It is therefore necessary to specify f_h, $n\tau_0$ and n or τ_0 for measurements of

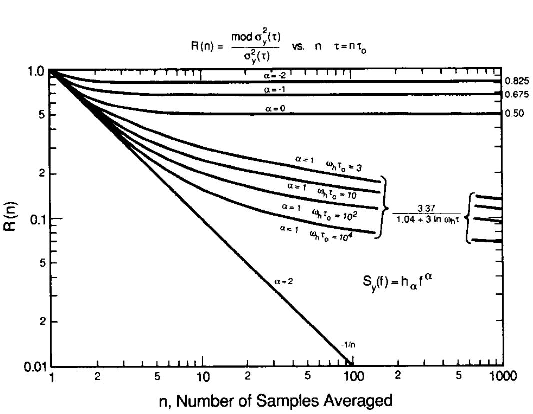

Fig. 8. Ratio of mod $\sigma_y^2(\tau)^2$ to $\sigma_y(\tau)^2$ as a function of n, the number of points averaged to obtain mod $\sigma_y(\tau)$ in the limit that $2\pi f_h \tau_0$ is much larger than 1. The measurement time $\tau = n\tau_0$, where τ_0 is the minimum data interval.

mod $\sigma_y(n\tau_0)$ whereas only f_h and $n\tau_0$ are required for $\sigma_y(n\tau_0)$. The effect of noise types $\alpha = 1$ and 2 on the determination of average frequency or time in practical sources can be eliminated by making n large enough or equivalently, τ_0 small enough. Under this condition, mod $\sigma_y(n\tau_0)$ becomes independent of n for n greater than some minimum value which depends on the ratio of $\alpha = 1$ and 2 noise to $\alpha = -1, -2$ and -3 noise. This extremely important result, first expounded by Bernier [34] in 1987, is now used to improve the determination of time interval and/or frequency in a number of precision applications.

VII. Calculation of $\sigma_y^2(\tau)$ and mod $\sigma_y^2(\tau)$ from the Fourier Frequency Domain

Cutler and Searle derived the calculation of $\sigma_y(\tau)$ from $S_y(f)$ [4]:

$$\sigma_y^2(\tau) = 2\int_0^\infty S_y(f)\frac{\sin^4(\pi f\tau)}{(\pi f\tau)^2}\,df. \tag{13}$$

Mod $\sigma_y(\tau)$ can be calculated from the Fourier frequency domain using [21], [26]–[28], [31], [33]

$$\text{mod } \sigma_y^2(n\tau_0) = 2\int_0^\infty S_y(f)\frac{\sin^6(\pi f n\tau_0)}{(\pi f n^2\tau_0)^2\sin^2(\pi f\tau_0)}df. \tag{14}$$

The integrals in (13) and (14) can be calculated analytically for a number of simple cases [26], [31]; however, the general case is most easily evaluated numerically [27], [28]. For both integrals it is necessary to specify the value and shape of the low-pass filter for noise types with $\alpha > 0$. The most common shapes are the infinitely sharp low-pass filter with cut off frequency f_h, and a single pole filter of equivalent noise bandwidth f_h. The results for $\alpha = 2$ are particularly dependent on the shape of the low-pass filter. See Figs. 4–9.

More generally, any variance is related to the spectral density $S_y(f)$ by an integral mathematical relationship (Section VIII, (20)) wherein the transfer function $H(f)$ is a Fourier transform of a stepwise function associated with the measurement response in the time domain [3]. It can be shown, and this is the key point, that the general shape, degree of selectivity, and the behavior at very low Fourier frequencies of the transfer function give a good understanding of the specific properties of the related variance. The transfer function approach has also been used to develop new types of experimental measurement systems as will be shown in Section VIII. Much of this development has been made possible by recent advances in high speed digital processing and microprocessors.

A. Application to Sinusoidal FM:

Even the best sources are frequency modulated by unwanted sinusoidal signals. Although the above stability measures were developed to deal with random processes, sinusoidal instabilities do have an influence on them.

1) Fourier Frequency Domain: If

$$y(t) = \frac{\Delta\nu_0}{\nu_0}\sin(2\pi f_m t) \tag{15}$$

then the spectral density $S_y(f)$ contains a discrete line at the modulating frequency f_m:

$$S_y(f) = \frac{1}{2}\left(\frac{\Delta\nu_0}{\nu_0}\right)^2\delta(f - f_m) \tag{16}$$

where δ is the Dirac delta function. Looking for discrete lines in spectral densities provides a convenient and powerful means to identify periodic variations. Their presence usually does not interfere with the identification of the slopes due to random noise. The substitution of (16) in (13) yields

$$\sigma_y^2(\tau) = \left(\frac{\Delta\nu_0}{V_0}\right)^2\frac{\sin^4(\pi f_m\tau)}{(\pi f_m\tau)^2}. \tag{17}$$

Substitution of (16) into (14) yields

$$\text{mod } \sigma_y^2(n\tau_0) = \left(\frac{\Delta\nu_0}{\nu_0}\right)^2\frac{\sin^6(\pi f_m n\tau_0)}{n^2(\pi f_m n\tau_0)^2\sin^2(\pi f_m\tau_0)}. \tag{18}$$

Thus the effect of sinusoidal FM in both cases is 0 when τ equals the modulation period $T_m = f_m^{-1}$ or one of its multiples, since the modulating signal is completely averaged away.

The largest value of mod $\sigma_y(\tau)$ due to sinusoidial FM occurs when τ is near $T_m/2$ or one of its odd multiples [9]. Mod $\sigma_y(n\tau_0)$ falls n times faster than $\sigma_y(n\tau_0)$ for $\pi f_m n\tau_0 >> \pi$ for sinusoidal FM. As a practical consequence, when caution has not been exercised about the relation between T_m and the experimental values of τ, some scatter of the data results because of the oscillating behavior of (17) and (18). This scatter is added to the contribution due to the random noise(s) present as illustrated in Fig. 9.

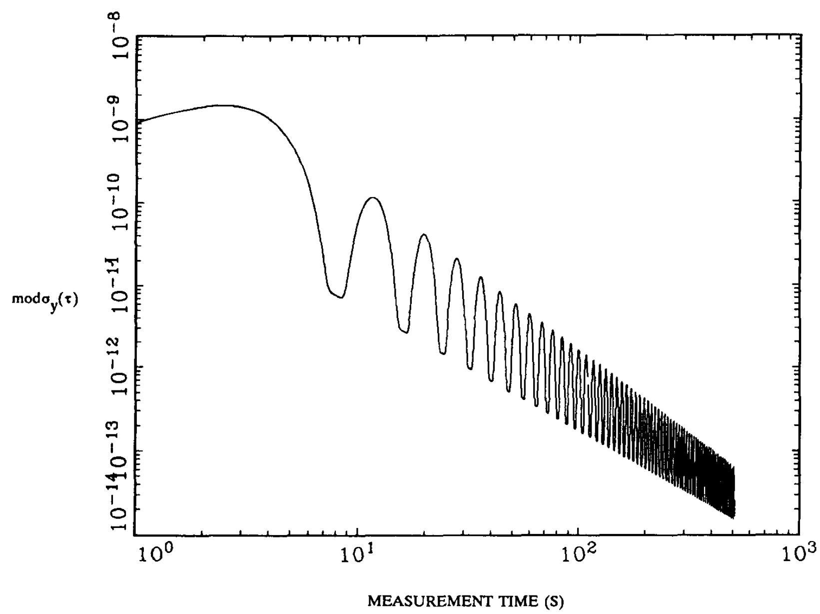

Fig. 9. Mod $\sigma_y(\tau)$ versus τ for the case of white phase noise ($S_y(f) = 4 \times 10^{-19} f^2, f_h = 1$, and frequency modulation with $S_y(f) = 3 \times 10^{-18}\delta(f - 0.125)$.

B. Application to Linear Frequency Drift:

Equations (13) and (14) are not useful for evaluating $\sigma_y^2(\tau)$ when linear frequency drift exists (i.e., $y(t) = d_1 t$) since no tractable model seems to exist for $S_y(f)$ in this case.

Direct calculation in the time domain from (10) and (12) yields

$$\sigma_y(\tau) = \frac{d_1}{\sqrt{2}}\tau \quad \text{and} \quad \text{mod }\sigma_y(\tau) = \frac{d_1}{\sqrt{2}}\tau. \tag{19}$$

Thus, linear frequency drift yields a τ^{+1} law for both $\sigma_y(\tau)$ and mod $\sigma_y(t)$.

VIII. Other Measures of Frequency Stability

A number of other measures have been proposed and used during the past 20 years. Each one possesses some advantages and limitations compared to the well-established ones. The Hadamard variance [13] has been developed for making high resolution spectral analysis of $y(t)$ from measurements of $\overline{y}_k$; that is, the frequency domain parameter $S_y(f)$ is estimated from data provided by digital counters. The high-pass variance [14] has been developed through the "transfer function approach" wherein a variance is defined by the transfer function $H(f)$ appearing in the general relationship

$$\sigma^2(\tau) = \int_0^\infty S_y(f)|H(f)|^2\, df. \tag{20}$$

It can be shown that an estimate of the Allan variance is provided by high-pass filtering the demodulated phase noise, without the need for counting techniques. A bandpass variance has also been introduced [14] to solve the problem of separating $\alpha = 1$ form $\alpha = 2$, and yields a slope of τ^{-1} for $\alpha = 1$ and $\tau^{-3/2}$ for $\alpha = 2$. A "filtered Allan variance" has also been used to separate the various noise types [29]. A modified three-adjacent-sample variance denoted as $\sum_y(\tau)$ has also been proposed [3] in the framework of a more general finite-time frequency control method [46]. The new parameter, whose experimental estimation by counting techniques is not very complicated, has the same general behavior as $\sigma_y(\tau)$ for power-law spectral densities, except that it converges for both $\alpha = -3$ and $\alpha = -4$ noise types. $\sum_y(\tau)$ varies as τ for $\alpha = 2$ and $\tau^{3/2}$ for $\alpha = 4$ noise types.

Another point of view was proposed in [47] which may appear more useful than $\sigma_y(\tau)$ for certain kinds of prediction problems. It relies on a two-sample variance (with nonadjacent samples) which is studied versus T for a fixed value of τ (not versus τ as in the other measures). The structure function approach [15], [16] plays a unifying role in the sense that most of the time-domain parameters appear as particular cases of the general structure function concept. This concept finds application in the determination of polynomial frequency drifts. A comparison of these and other approaches can be found in [3], [10], [13]–[16], [33]–[44].

IX. Conclusion

We have briefly described some of the developments and features of frequency stability measures, both in the Fourier frequency domain and in the time-domain. Spectral densities play a key role in the sense that all the time-domain measures may be deduced from them, whereas general inversion of the integral formulas is usually very difficult if not impossible. It has been shown that the concept of the transfer function allows one to understand clearly the advantages and limitations of each parameter.

The primary focus has been the Allan or two sample variance and the modified Allan variance because they are by far the most commonly used. They provide useful well-behaved measures for all random noise types found in precision oscillators and equipment used in signal processing. They are, however, general purpose measures and other measures may be more useful in specific cases [3]. Complete mathematical descriptions of these measures can be found in the included references. The contribution of many individuals must be recognized together with the initiating and coordinating role of organizations of NASA, NBS (now NIST), IEEE, and CCIR.

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