I. Introduction to Numerical Methods

A. Conventions

1. In optics and photonics, both physicists and engineers work. The conventions are a mix of both.
2. Additionally, computer algorithms are usually based on standard computer science/mathematics conventions in packages such as Matlab, NAG, IMSL, Linpack, Eispack, etc.

! Thus, workers who are doing computing must be aware of all three conventions and know how to translate among them

3. This difference is found in my group's work
   a. Most of our papers are based on the <u>physics</u> convention
   b. OCS is based on the math/CS convention, which is close to the engineering convention.

4. The basic difference
   Let $u(x, t)$ be a waveform that is the sum of a finite number of waves. In that case:

a. Physics convention:

$$u(z,t) = \sum_{n=1}^{N} A_n \cos(k_n z - \omega_n t + \phi_n)$$

$k_n \equiv$ wavenumber, $\omega_n =$ frequency, $z =$ distance, $t =$ time, $A_n =$ amplitude, $\phi_n =$ phase. $N =$ positive integer.

b. Mathematics convention; Note that this is equivalent to backward propagation

$$u(z,t) = \sum_{n=1}^{N} A_n \cos(\omega_n t + k_n z + \phi_n)$$

c. Engineering convention [Put also $\omega_n = 2\pi\nu_n$, $\beta_n = 2\pi/\lambda_n$]

$$u(z,t) = \sum_{n=1}^{\infty} A_n \cos\left(2\pi\nu_n t - \frac{2\pi}{\lambda_n} z + \phi_n\right)$$

The sign change in going from the physics convention to the engineering/mathematics/CS may appear trivial, but it has many important ramifications.

It is usual to express these sums using complex exponentials

a. Physics convention

$$u(z,t) = \sum_{n=1}^{N} a_n \exp[i(k_n z - \omega_n t)]$$

b. Mathematics convention

$$u(z,t) = \sum_{n=1}^{N} a_n \exp[i(\omega_n t + k_n z)]$$

c. Engineering convention

$$u(z,t) = \sum_{n=1}^{N} a_n \exp\left[2\pi j\left(\nu_n t - \frac{z}{\lambda_n}\right)\right]$$

$i = \sqrt{-1}$ (mathematicians and physicists)
$j = \sqrt{-1}$ (engineers)

6. Continuous limit

a. Physics convention

Sometimes this factor is left out

$$u(z,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\omega) \exp\left[i(kz - \omega t)\right]$$

(Why the factor of 2π? You will see)

b. Mathematics convention

$$u(z,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\omega) \exp\left[i(\omega t + kz)\right]$$

c. Engineering convention

$$u(z,t) = \int_{-\infty}^{\infty} d\nu \, A(\nu) \exp\left[2\pi j\left(\nu t - \frac{1}{\lambda} z\right)\right]$$

Factor of 2π allows us to directly connect $A(\omega)$ and $A(\nu)$: $A_\omega(\omega) = A_\nu(2\pi\nu)$
The functions are not the same!

7. Dispersion relations

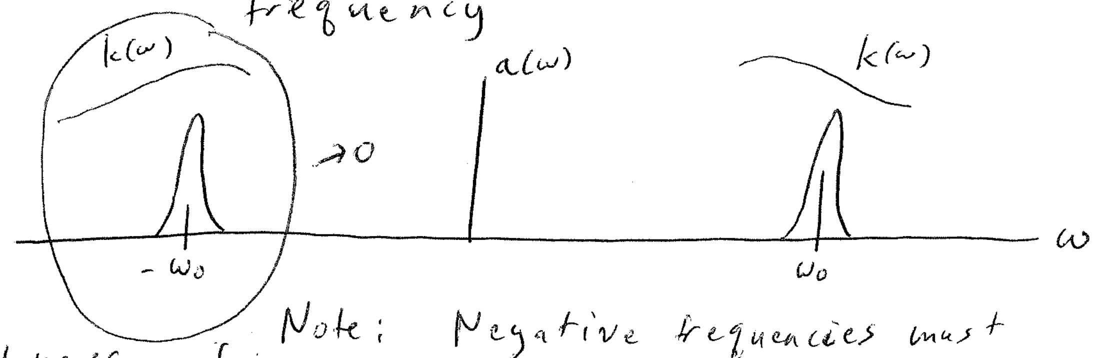
a. The relationship between k and ω is fixed by the dispersion relationship: $k = k(\omega)$, which is a property of the medium

In vacuum: $k(\omega) = \omega / c$

More generally: $k(\omega) = \omega\, n(\omega) / c$

In glass fibers, $n(\omega)$ is a slowly varying function in the neighborhood of $\lambda_0 = 1.5\ \mu$ and $n(\omega = 2\pi c/\lambda_0) \simeq 1.5$. Note: λ_0 is the wavelength <u>outside the glass</u> and equals $\omega / 2\pi c$!

b. In communications signals, the signal bandwidth is always small compared to the carrier frequency



However, we normally cut off this part and consider only positive frequencies.

Note: Negative frequencies must be present if $u(z,t)$ is real.

If $k(\omega)$ varies slowly, then we can Taylor expand $k(\omega)$

We must put it back when considering nonlinearity

$$\overset{\equiv k_0}{k(\omega) - k(\omega_0)} = \overset{\equiv k_0'}{\left.\frac{dk}{d\omega}\right|_{\omega_0}} (\omega - \omega_0) + \frac{1}{2} \overset{\equiv k_0''}{\frac{d^2k}{d\omega^2}} (\omega - \omega_0)^2$$
$$+ \frac{1}{6} \overset{\equiv k_0'''}{\frac{d^2k}{d\omega^2}} (\omega - \omega_0)^2 + \ldots$$

We can remove the carrier frequency by writing

(1) Physics convention:

$$u(z,t) = \bar{u}(z,t) \exp[i k_0 z - i\omega_0 t)$$

(2) Mathematics convention

$$u(z,t) = \bar{u}(z,t) \exp(i\omega_0 t + i k_0 z)$$

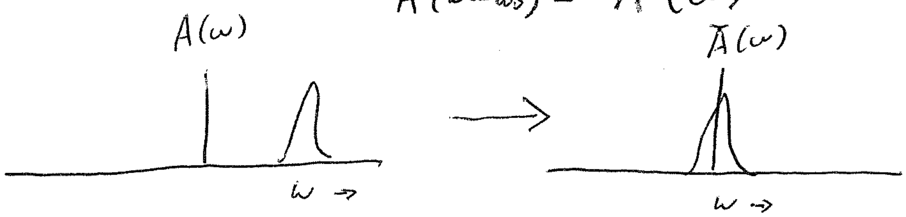
(3) Engineering convention

$$u(z,t) = \bar{u}(z,t) \exp(j\omega_0 t - j\beta_0 z)$$

[Replace ν, λ with ω, β]

We now translate $A(\omega)$: $\bar{A}(\omega)$

$$\bar{A}(\omega - \omega_0) = A(\omega)$$



C. We now find: physics convention

$$\bar{u}(z,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{A}(\omega - \omega_0) \exp\{i[k(\omega) - k_0] z - i(\omega - \omega_0) t\}$$

Letting $\Omega = \omega - \omega_0$, $\quad K(\Omega) = k(\omega_0 + \Omega) - k_0,$
we find $\qquad \simeq K_0'\Omega + \frac{1}{2}K_0''\Omega^2 + \frac{1}{6}K_0'''\Omega^3 + \ldots$

$$\bar{u}(z,t) = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \bar{A}(\Omega) \exp\{i[K(\Omega)z - \Omega t]\}$$

d. We can now derive a propagation equation:

$$\frac{\partial \bar{u}(z,t)}{\partial z} = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} [iK(\Omega)] \bar{A}(\Omega) \exp\{i[K(\Omega)z - \Omega t]\}$$

$$= \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \left[iK_0'\Omega + \frac{i}{2}K_0''\Omega^2 + \frac{i}{6}K_0'''\Omega^3\right] \cdot \exp\{i[K(\Omega)z - \Omega t]\}$$

$$= -K_0' \frac{\partial \bar{u}(z,t)}{\partial t} - \frac{i}{2}K_0'' \frac{\partial^2 \bar{u}(z,t)}{\partial t^2} + \frac{1}{6}K_0''' \frac{\partial^3 \bar{u}(z,t)}{\partial t^3} + \ldots$$

or:

$$i\frac{\partial \bar{u}(z,t)}{\partial z} + iK_0' \frac{\partial \bar{u}(z,t)}{\partial t} - \frac{1}{2}K_0'' \frac{\partial^2 \bar{u}(z,t)}{\partial t^2} - \frac{i}{6}K_0''' \frac{\partial^3 \bar{u}(z,t)}{\partial t^3} + \ldots = 0$$

Dropping the bars and returning to k

$$i\frac{\partial u(z,t)}{\partial z} + ik_0' \frac{\partial u(z,t)}{\partial t} - \frac{1}{2}k_0'' \frac{\partial^2 u(z,t)}{\partial t^2} - \frac{i}{6}k_0'' \frac{\partial^3 u(z,t)}{\partial t^3} + \ldots = 0$$

e. We now subtract the group velocity motion

$$t' = t - k_0' z; \quad z' = z$$

$$i \frac{\partial u(z,t)}{\partial z} - \frac{1}{2} k_0'' \frac{\partial^2 u(z,t)}{\partial t^2} - \frac{i}{6} k_0''' \frac{\partial^3 u(z,t)}{\partial t^3} + \ldots = 0$$

where we have dropped the primes.

f. Other conventions:

(1) Mathematics convention

$$i \frac{\partial u(z,t)}{\partial z} - \frac{1}{2} k_0'' \frac{\partial^2 u(z,t)}{\partial t^2} + \frac{i}{6} k_0''' \frac{\partial^3 u(z,t)}{\partial t^3} + \ldots = 0$$

(2) Engineering convention β_0'''

$$j \frac{\partial u(z,t)}{\partial z} + \frac{1}{2} \beta_0'' \frac{\partial^2 u(z,t)}{\partial t^2} - \frac{j}{6} \frac{\partial^3 u(z,t)}{\partial t^3} + \ldots = 0$$

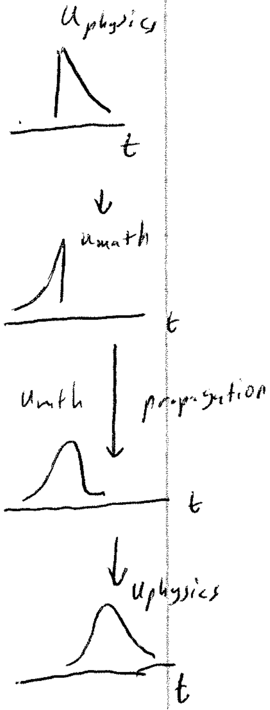
Insert on transformation rels.

8. Polarization conventions

a. In a birefringent medium — and all fibers have some birefringence — there are two dispersion relations $k_+(\omega)$ and $k_-(\omega)$, (which differ by about 1 part in 10^6 in fibers)

1, 7A

7A. Transforming between conventions



a. $u_{physics}(z, t) = u_{math}(z, -t)$

It is just a sign reversal of time
In OCS, you would make this change at the input and output

Note: $a_{physics}(z, \omega) = \int_{-\infty}^{\infty} dt\, u_{physics}(z, t) \exp(i\omega t)$

$$= \int_{-\infty}^{\infty} dt\, u_{math}(z, -t) \exp(i\omega t) \qquad t \to -t'$$

$$= \int_{-\infty}^{\infty} dt\, u_{math}(z, t') \exp(-i\omega t')$$

$$= a_{math}(z, \omega)$$

There is no change in the spectrum because in the physics convention one defines the Fourier transform with the opposite sign from the math and physics conventions

b. $u_{eng}(z, t) = u^*_{physics}(z,t) = u^*_{math}(z, -t)$

$a_{eng}(z, \omega) = a^*_{physics}(z, \omega) = a^*_{math}(z, \omega)$

because of sign change in the definition of the Fourier transform

7B. Who uses what convention?

<u>Math convention</u>: OCS at present
This convention is consistent with standard math packages like Matlab

<u>Physics convention</u>: Agrawal, Born and Wolf, Our research group

<u>Engineering convention</u>: Haus (always), Kazovsky, Benedetto, and Wilner (except chap. 7 – watch out – compare chaps. 6 and 7)

Most of the engineering literature.

In the x and y direction with the physics convention

$$u_x = A_+ \cos\theta \exp[i(k_+ z - \omega t) + i\phi_1] + A_- \sin\theta \exp[i(k_- z - \omega t) + i\phi_2]$$

$$u_y = -A_+ \sin\theta \exp[i(k_+ z - \omega t) - i\phi_2] + A_- \cos\theta \exp[i(k_- z - \omega t - i\phi_1]$$

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \cos\theta e^{i\phi_1} & \sin\theta e^{i\phi_2} \\ -\sin\theta e^{-i\phi_2} & \cos\theta e^{-i\phi_1} \end{pmatrix} \begin{pmatrix} A_+ e^{ik_+ z} \\ A_- e^{ik_- z} \end{pmatrix}$$

Let $\beta = (\phi_1 + \phi_2)/2$, $\gamma = (\phi_1 - \phi_2)/2$ | we let $\gamma = 0$ $\to$ constant offset

$$S_0 = |u_x|^2 + |u_y|^2 = A_+^2 + A_-^2$$

$$S_1 = |u_x|^2 - |u_y|^2 = (A_+^2 - A_-^2)\cos 2\theta + 2A_+A_- \sin 2\theta \cos[\underbrace{(k_+ - k_-)}_{\Delta k} z + 2\gamma]$$

$$S_2 = 2\,\mathrm{Re}(u_x u_y^*) = -(A_+^2 - A_-^2)\sin 2\theta \cos(2\beta) - 2A_+A_- \sin(2\beta)\sin(\Delta k z + 2\gamma) + 2A_+A_- \cos(2\beta)\cos 2\theta \cos(\Delta k z + 2\gamma)$$

$$S_3 = 2\,\mathrm{Im}(u_x u_y^*) = -(A_+^2 - A_-^2)\sin 2\theta \sin(2\beta) + 2A_+A_- \cos(2\beta)\sin(\Delta k + 2\gamma) + 2A_+A_- \sin(2\beta)\cos 2\theta \cos(\Delta k z + 2\gamma)$$

Which is a circle on a sphere of radius $A_1^2 + A_2$

b. It is normal to refer to a polarization state as right-handed when it advances clockwise when approaching the observer

That occurs when $S_3 > 0$

Conversely, light is referred to as left-handed when it advances counter-clockwise when approaching the observor, which occurs when $S_3 < 0$.

c. In the engineering convention, just the opposite is true. However, in a recent article by Gordon and Kogelnik, just the opposite is true, they change the convention, so that the sign of S_3 is changed.

d. The mathematics convention is more subtle. Perhaps the best way to think of it is as equivalent to the physics convention once we set $t \to -t$.