Solutions to the optical cascading equations

S. Lafortune and P. Winternitz
Centre de Recherches Mathématiques, Université de Montréal, Case Postale 6128, Succursale Centre-ville, Montréal, Québec H3C 3J7, Canada

C. R. Menyuk
Department of Electrical Engineering, University of Maryland, Baltimore, Maryland 21228-5398

(Received 8 January 1998)

Group theoretical methods are used to study the equations describing $\chi^{(2)}:\chi^{(2)}$ cascading. The equations are shown not to be integrable by inverse scattering techniques. On the other hand, these equations do share some of the nice properties of soliton equations. Large families of explicit analytical solutions are obtained in terms of elliptic functions. In special cases, these periodic solutions reduce to localized ones, i.e., solitary waves. All previously known explicit solutions are recovered, and many additional ones are obtained.
[S1063-651X(98)06207-2]

PACS number(s): 42.65.Tg, 03.40.Kf, 02.20.$-$a

I. INTRODUCTION

Materials with a significant $\chi^{(3)}$ nonlinearity exhibit solitonlike beam or pulse propagation. These materials can be modeled by the nonlinear Schrödinger equation and its variants [1], which are known to be integrable and possess exact soliton solutions.

In the case of $\chi^{(2)}$ materials, it has been possible to produce solitary waves through $\chi^{(2)}:\chi^{(2)}$ cascading [2]. This effect occurs in parametrically coupled fields with quadratic nonlinearities in which the nonlinearities in the fundamental and first harmonic fields interact in a way that mimics the $\chi^{(3)}$ nonlinearity. Moreover, several particular cases of solitary wave solutions of the system describing this phenomenon have been obtained numerically and analytically ([3–9]).

The solitons of $\chi^{(3)}$ type are known to belong to families that include periodic solutions. It is essential to know if this is the case for the solitary waves of the $\chi^{(2)}$ type, mainly because the system describing them is not integrable. We do not know what properties of integrable systems these solitary wave solutions possess.

Moreover, the periodic solutions containing these solitary waves as limiting cases can be important in their own right, especially if they are stable. For instance, they may propagate as a background to solitary wave signals.

In this paper we find families of periodic solutions expressed in terms of elliptic functions. As special limiting cases, we obtain elementary trigonometric solutions and also localized solitary waves (including all those found earlier). We mention that not all of the solitary waves are stable ([10–12]) and that the stability of periodic solutions needs a separate study.

In Sec. II we present the $\chi^{(2)}:\chi^{(2)}$ cascading equations, and find their Lie point symmetry group. We then reduce the system to a set of four coupled real ordinary differential equations for traveling wave solutions. These reduced equations are further studied in Sec. III. We show that in general the equations do not have the Painlevé property [13,14] so the cascading equations are not integrable. We do, however, show that well behaved analytical solutions exist in special cases. Explicitly, elliptic function solutions are presented in Sec. IV together with their elementary function limits. Some further special cases are discussed in Sec. V. Conclusions are drawn in the final Sec. VI.

II. CASCADING EQUATIONS AND THEIR REDUCTION

Let us write the cascading equations in the normalized forms [3]

$$ia_{1t}-\frac{r}{2}a_{1xx}+a_1^*a_2=0,$$

$$ia_{2t}-\beta a_2-i\delta a_{2x}-\frac{\alpha}{2}a_{2xx}+a_1^2=0, \tag{1}$$

where a_1 and a_2 are the normalized (complex) envelopes of the fundamental and second harmonic waves, respectively. They are functions of two variables, namely, t, the normalized distance along the wave guide, and x, the normalized transverse coordinate. The constants involved are $r=\pm 1$ and α, β, and δ (all reals). Physically, α is given by minus the ratio of the wave numbers of the fundamental, and its first harmonic which nearly equals -0.5. This ratio becomes arbitrary when considering temporal structures, but these are difficult in practice to observe [3]. The quantity β corresponds to the normalized wave number mismatch. It may be written $\beta=k_1\eta^2\Delta k$, where $\Delta k=2k_1-k_2$ is the actual physical mismatch and η is a parameter that characterizes the size of a single beam which is 10 to 50 μm in typical experiments [2]. One then finds that β is typically in the range -50 to 50. The parameter δ corresponds to a normalized walkoff coefficient, and may be written $\delta=k_1\eta\rho$, where ρ is the actual walkoff angle. In typical experiments, δ is in the range -2 to 2.

To obtain explicit solutions we use the method of symmetry reduction, i.e., use Lie point symmetries to reduce Eqs.

(1) to ordinary differential equations. We first replace Eq. (1) by four real equations, putting

$$a_k(x,t)=R_k(x,t)e^{i\phi_k(x,t)}, \quad k=1,2,$$

$$0\leqslant R_k<\infty, \quad 0\leqslant \phi_k<2\pi, \tag{2}$$

and apply a standard algorithm to find their symmetry group [15–17]. In the generic case, when all constants in the equation are arbitrary, the symmetry group of the equations is just three dimensional, consisting of translations in x and t and correlated shifts of the phase $\phi_1\rightarrow\phi_1+\lambda$ and $\phi_2\rightarrow\phi_2+2\lambda$. The Lie algebra of the symmetry group has a basis consisting of the operators

$$P_1=\partial_x, \quad P_0=\partial_t, \quad W=\partial_{\phi_1}+2\partial_{\phi_2}. \tag{3}$$

We shall consider solutions invariant under the subgroup generated by

$$X=vP_1+\omega P_0+(\kappa_1\omega-\omega_1 v)W, \tag{4}$$

where v, ω, κ_1, and ω_1 are real constants.

The invariant solutions then have the forms

$$a_1(\xi)=R_1(\xi)e^{i\phi_1(\xi)}e^{i(\kappa_1 t-\omega_1 x)},$$

$$a_2(\xi)=R_2(\xi)e^{i\phi_2(\xi)}e^{2i(\kappa_1 t-\omega_1 x)},$$

$$\xi=\omega x-vt. \tag{5}$$

The four real functions R_1, R_2, ϕ_1, and ϕ_2 satisfy the following system of coupled ordinary differential equations:

$$-\frac{r}{2}\omega^2(2R_1'\phi_1'+R_1\phi_1'')+(r\omega\omega_1-v)R_1'+R_1R_2\sin\,\phi=0, \tag{6}$$

$$-\frac{\alpha}{2}\omega^2(2R_2'\phi_2'+R_2\phi_2'')+(2\alpha\omega\omega_1-v-\delta\omega)R_2'-R_1^2\sin\phi=0, \tag{7}$$

$$-\frac{r}{2}\omega^2(R_1''-R_1\phi_1'^2)+(v-r\omega\omega_1)R_1\phi_1'+\left(\frac{r}{2}\omega_1^2-\kappa_1\right)R_1+R_1R_2\,\cos\,\phi=0, \tag{8}$$

$$-\frac{\alpha}{2}\omega^2(R_2''-R_2\phi_2'^2)-(2\alpha\omega\omega_1-v-\delta\omega)R_2\phi_2'+(2\alpha\omega_1^2-2\kappa_1-\beta-2\delta\omega_1)R_2+R_1^2\cos\,\phi=0, \tag{9}$$

$$\phi\equiv\phi_2-2\phi_1.$$

It is the system of equations (6)–(9) that we wish to solve.

III. ANALYSIS OF THE REDUCED EQUATIONS

A. Phase locked solutions

The system of equations (6)–(9) is in general quite difficult to decouple and solve. It is greatly simplified if we impose a supplementary restriction on the phases, namely,

$$\phi=k\pi, \quad k\in\mathbb{Z},$$

i.e.,

$$\sin\,\phi=0, \quad \cos\,\phi=\epsilon, \quad \epsilon^2=1. \tag{10}$$

We shall call such solutions ''phase locked solutions.'' Let us first assume

$$r\alpha\omega\neq 0, \tag{11}$$

and simplify notations, putting

$$A=\frac{r\omega\omega_1-v}{r\omega^2}, \quad B=\frac{2\alpha\omega\omega_1-v-\delta\omega}{\alpha\omega^2},$$

$$C=2\frac{2\alpha\omega_1^2-2\kappa_1-2\delta\omega_1-\beta}{\alpha\omega^2}, \quad D=\frac{r\omega_1^2-2\kappa_1}{r\omega^2},$$

$$M_1=-\frac{2N_1}{r\omega^2}, \quad M_2=-\frac{2N_2}{\alpha\omega^2}. \tag{12}$$

Equations (6)–(9) can be rewritten as

$$\phi_1'=A+\frac{M_1}{R_1^2}, \tag{13}$$

$$\phi_2'=B+\frac{M_2}{R_2^2}, \tag{14}$$

$$R_1''-R_1\phi_1'^2+2AR_1\phi_1'-DR_1-\frac{2\epsilon}{r\omega^2}R_1R_2=0, \tag{15}$$

$$R_2''-R_2\phi_2'^2+2BR_2\phi_2'-CR_2-\frac{2\epsilon}{\alpha\omega^2}R_1^2=0. \tag{16}$$

B. Case $M_1M_2\neq 0$

The phase locking condition (10) imposes a relation between R_1 and R_2, namely,

$$B+\frac{M_2}{R_2^2}=2A+\frac{2M_1}{R_1^2}, \tag{17}$$

so that the system (13)–(17) is overdetermined. Expressing R_1, ϕ_1', and ϕ_2' in terms of R_2 and substituting into Eqs. (15) and (17), we obtain two second order ordinary differential equations for R_2. These turn out to be compatible only for R_1 and R_2 constant. This case will be considered separately in Sec. V below.

C. Case $M_1M_2=0$

Again R_1 and R_2 are constant, unless we have

$$M_1=M_2=0. \tag{18}$$

Let us investigate case (18). We have

$$\phi_1' = A, \quad \phi_2' = B = 2A. \tag{19}$$

We put

$$A_0 = A^2 - D, \quad B_0 = -\frac{2\epsilon}{r\omega^2}, \quad C_0 = B^2 - C, \quad D_0 = -\frac{2\epsilon}{\alpha\omega^2}, \tag{20}$$

and obtain a system of two ordinary differential equations

$$R_1'' + A_0 R_1 + B_0 R_1 R_2 = 0,$$

$$R_2'' + C_0 R_2 + D_0 R_1^2 = 0. \tag{21}$$

This system is not overdetermined since constraint (17) is now simply the condition (19) on the constants, i.e.,

$$2\alpha v - rv - \delta r\omega = 0. \tag{22}$$

Eliminating R_2 from Eq. (21), we obtain a fourth order equation for R_1, namely,

$$\left(\frac{R_1''}{R_1}\right)'' + C_0 \frac{R_1''}{R_1} - B_0 D_0 R_1^2 + A_0 C_0 = 0. \tag{23}$$

If the original Eqs. (1) are integrable, then Eq. (23) should have the Painlevé property, i.e., have no movable singularities other than poles [13,14]. An algorithmic test exists [13,18], establishing certain properties of an equation, necessary for it to have the Painlevé property.

Thus, the general solution of Eq. (23) must allow an expansion in the neighborhood of any singular point of the form

$$R_1 = \sum_{k=0}^{\infty} a_k (\xi - \xi_0)^{k+p}, \tag{24}$$

with p a negative integer, $a_0 \neq 0$, and three of the coefficients a_k arbitrary. Then R_1 has a good chance of representing the general solution of Eq. (23), depending on four arbitrary constants (one of them being ξ_0, the position of the pole). The values of k for which a_k are arbitrary (i.e., are not determined by a recursion relation) are called ''resonance'' values.

Substituting the expansion (24) into Eq. (23), we find $p = -2$, $a_0^2 = 36/(B_0 D_0)$, and the resonance values

$$r = -1, \ \ 6, \ \ (5 \pm i\sqrt{23})/2. \tag{25}$$

Thus, we have only one nonnegative integer, namely, $r = 6$, rather than the three needed. An analysis of the obtained recursion relations shows that $a_0, \ldots, a_5$ are fully determined; a_6 is indeed free, and can be chosen arbitrarily. Then a_7 and all the higher terms are fully determined in terms of ξ_0 and a_6 (and of course A_0, B_0, C_0, and D_0). Thus Eq. (23) does not have the Painlevé property, and the cascading equations (1) are not integrable.

The Painlevé analysis does, however, indicate that families of ''well behaved'' solutions should exist (i.e., single valued in the neighborhood of their movable singularities), depending on one or two free parameters rather than on four. We shall find such solutions below.

An alternative procedure is to solve Eq. (21), again under the condition (22), for R_2. We obtain the ordinary differential equation

$$2(R_2'' + C_0 R_2)(R_2'''' + C_0 R_2'') - (R_2''' + C_0 R_2')^2 + 4(A_0 + B_0 R_2) \times (R_2'' + C_0 R_2)^2 = 0. \tag{26}$$

A Painlevé analysis of Eq. (26) leads to the same conclusion as that of Eq. (23).

D. Introduction of the elliptic function equation

Let us look for solutions $R_2(\xi)$ satisfying Eq. (26), and also the elliptic function equation

$$R_2'^2 = \gamma_4 R_2^4 + \gamma_3 R_2^3 + \gamma_2 R_2^2 + \gamma_1 R_2 + \gamma_0. \tag{27}$$

The compatibility of Eqs. (26) and (27) implies

$$\gamma_4 = 0, \tag{28}$$

and imposes six relations between the constants in Eqs. (26) and (27). These allow for the following solutions.

$$(1) \quad \gamma_3 \neq 0, \quad C_0(C_0 - A_0) \neq 0 \tag{29}$$

The constants γ_μ in Eq. (27) are completely specified:

$$\gamma_3 = -\frac{2B_0}{3}, \quad \gamma_2 = C_0 - 2A_0,$$

$$\gamma_1 = \frac{(C_0 + 2A_0)(C_0 - A_0)}{B_0},$$

$$\gamma_0 = \frac{(C_0 + 2A_0)^2 (C_0 - A_0)}{6B_0^2}. \tag{30}$$

The other amplitude $R_1(\xi)$ is given directly by the expression

$$R_1^2 = \frac{1}{D_0}\left(B_0 R_2^2 + 2(A_0 - C_0)R_2 + \frac{(A_0 - C_0)(2A_0 + C_0)}{2B_0}\right), \tag{31}$$

and we must require R_1^2 to satisfy

$$R_1^2 \geq 0 \tag{32}$$

in the entire range of values of R_2, a condition to be analyzed below.

Notice that Eq. (27) will have solutions depending on just one parameter, an integration constant, since the coefficients γ_μ are fixed in terms of A_0, B_0, C_0, and D_0. The only constraints on the constants in the original equations (1) r, α, β, and δ, and those introduced in the reduction procedure (5) namely, v, ω, κ_1, and ω_1 are Eqs. (11), (22), and (32).

$$(2) \quad \gamma_3 \neq 0, \quad C_0 = 0 \tag{33}$$

In this case, we obtain

$$\gamma_3 = -\frac{2B_0}{3}, \quad \gamma_2 = -2A_0, \quad \gamma_1 = -\frac{2A_0^2}{B_0}, \tag{34}$$

and γ_0 is arbitrary. Moreover, we have

$$R_1=\sqrt{\frac{B_0}{D_0}}\left(R_2+\frac{A_0}{B_0}\right). \tag{35}$$

Thus we obtain a two parameter family of solutions, the parameters being γ_0 and a constant arising in the integration of Eq. (27). The constraints on the coefficients are Eqs. (11), (22), and $C_0=0$, i.e.,

$$2\alpha v^2+\omega(2\kappa_1\omega+\beta\omega-2\omega_1 v)=0 \tag{36}$$

(we have $r^2=1$).

(3) $\gamma_3\neq 0,\quad C_0=A_0$ (37)

We then have

$$\gamma_3=-\frac{2B_0}{3},\quad \gamma_2=-A_0,\quad \gamma_1=0, \tag{38}$$

and γ_0 arbitrary. Again, Eq. (27) provides a two parameter family of solutions, and the constraint $C_0=A_0$ is

$$3\alpha v^2+2r\omega(\alpha-2r)(\omega_1 v-\kappa_1\omega)+2\beta\omega^2=0, \tag{39}$$

and we have

$$R_1=\sqrt{\frac{B_0}{D_0}}R_2. \tag{40}$$

(4) $\gamma_3=0$ (41)

In this case, Eq. (27) reduces to

$$R_2'^2=-C_0R_2+\gamma_0. \tag{42}$$

The solutions are

$$R_1=0 \tag{43}$$

$$R_2=\begin{cases}-\dfrac{C_0}{4}(\xi-\xi_0)^2+\mu, & C_0\neq 0\\ \mu(\xi-\xi_0), & C_0=0.\end{cases} \tag{44}$$

E. Phase locked solutions for $\alpha=0$

We return to Eqs. (6)–(9) for $\alpha=0$, $\sin\phi=0$, and $\cos\phi=\epsilon$. From Eq. (7), we see that $R_2\neq$const implies

$$v+\delta\omega=0. \tag{45}$$

Equation (6) can be integrated to give

$$\phi_1'=\frac{2N}{r\omega^2}\frac{1}{R_1^2}+\frac{r\omega\omega_1-v}{2}, \tag{46}$$

where N is an integration constant.

In order to have $R_1\neq 0$, we impose

$$2\kappa_1+\beta+2\delta\omega_1\neq 0 \tag{47}$$

and obtain from Eq. (9) that we have

$$R_1=[\epsilon(2\kappa_1+\beta+2\delta\omega_1)R_2]^{1/2}. \tag{48}$$

Finally, Eq. (8) implies that R_2 satisfies the elliptic function equation

$$(R_2')^2=-2B_0R_2^3-4A_0R_2^2+SR_2-\frac{16N^2}{\omega^4(2\kappa_1+\beta+2\delta\omega_1)^2}. \tag{49}$$

Since both N and S are arbitrary integration constants, Eq. (49) yields a three parameter family of solutions.

IV. SOLUTIONS IN TERMS OF ELLIPTIC FUNCTIONS AND THEIR LIMITING CASES

A. General comments on the elliptic function equation

In Sec. III we have obtained three equations of the type

$$R'^2=\beta_3R^3+\beta_2R^2+\beta_1R+\beta_0, \tag{50}$$

where β_i are real constants and $\beta_3\neq 0$. Putting

$$f=\beta_3R, \tag{51}$$

we obtain the equation

$$f'^2=f^3+\beta_2f^2+\beta_1\beta_3f+\beta_0\beta_3^2. \tag{52}$$

We introduce the roots f_1, f_2, and f_3 of the polynomial on the right hand side of Eq. (52), and rewrite this equation as

$$f'^2=(f-f_1)(f-f_2)(f-f_3). \tag{53}$$

The solutions of this equation can be expressed in terms of Jacobi elliptic functions [19] if the three roots are distinct. The case of multiple roots leads to solutions in terms of elementary functions.

Let us first consider the case of three real roots, and order them to satisfy $f_1\leq f_2\leq f_3$. We are only interested in real solutions.

(1) $f_1\leq f\leq f_2<f_3$

We obtain a finite periodic solution

$$f=(f_2-f_1)\mathrm{sn}^2(u,k)+f_1,$$

$$k^2=\frac{f_2-f_1}{f_3-f_1},\quad u=\frac{\sqrt{f_3-f_1}}{2}(\xi-\xi_0), \tag{54}$$

where ξ_0 is a real integration constant.

(2) $f_1\leq f\leq f_2=f_3$

A special (limiting) case of solution (54) is obtained for $f_2=f_3$, namely the solitary wave solution

$$f=(f_2-f_1)\tanh^2(u)+f_1, \tag{55}$$

with u as in Eq. (54).

(3) $f_1<f_2<f_3\leq f$

We obtain a singular periodic solution

$$f=(f_3-f_1)\frac{1}{\mathrm{sn}^2(u,k)}+f_1, \tag{56}$$

with u and k as in Eq. (54).

(4) $f_1=f_2<f_3\leqslant f$

For $f_1=f_2$, solution (56) reduces to an elementary periodic singular solution, namely,

$$f=(f_3-f_1)\frac{1}{\sin^2(u)}+f_1. \tag{57}$$

(5) $f_1<f_2=f_3\leqslant f$

For $f_2=f_3$, solution (56) reduces to a ''singular solitary wave,'' namely,

$$f=(f_3-f_1)\coth^2(u)+f_1. \tag{58}$$

(6) $f_1=f_2=f_3\leqslant f$

A triple root corresponds to a ''singular algebraic solitary wave''

$$f=\frac{4}{(\xi-\xi_0)^2}+f_1. \tag{59}$$

The case of one real and two mutually complex conjugated roots leads to singular periodic solutions.

(7) $f_1=a+ib,\quad f_2=a-ib,\quad f_3,a,b\in\mathbb{R},\quad b>0$

The solutions are

$$f=\frac{f_3+P+(f_3-P)\mathrm{cn}(u,k)}{1+\mathrm{cn}(u,k)}, \tag{60}$$

$$u=\sqrt{P}(\xi-\xi_0),\quad k^2=\frac{P+a-f_3}{2P},\quad P^2=(a-f_3)^2+b^2. \tag{61}$$

B. Explicit solutions of the cascading equations for $\alpha\neq 0$

(1) Case (29)
We have

$$\beta_2=C_0-2A_0,\quad \beta_1\beta_3=-\frac{2}{3}(C_0+2A_0)(C_0-A_0)$$

$$\beta_0\beta_3^2=\frac{2}{27}(C_0+2A_0)^2(C_0-A_0),\quad C_0\neq 0,\quad C_0\neq A_0 \tag{62}$$

in Eq. (52). Using Eq. (31), we have

$$R_2=-\frac{3}{2B_0}f,$$

$$R_1^2=\frac{9}{4B_0D_0}\left[f^2-\frac{4}{3}(A_0-C_0)f+\frac{2}{9}(A_0-C_0)(2A_0+C_0)\right]. \tag{63}$$

A globally defined solution exists if R_1^2 is positive for all values of f. We note that the roots of the polynomial defining R_1^2 are

$$f_{A,B}=\tfrac{1}{3}[2(A_0-C_0)\pm\sqrt{6C_0(C_0-A_0)}]. \tag{64}$$

Hence for $C_0(C_0-A_0)<0$ the function R_1^2 is sign definite, and can be chosen to be positive definite. For $C_0(C_0-A_0)>0$ a more careful analysis is required. Thus solution (54) provides a global solution if the roots satisfy one of the following relations:

$$f_A\leqslant f_1\leqslant f\leqslant f_2\leqslant f_B,\quad f_A\leqslant f_B\leqslant f_1\leqslant f\leqslant f_2,$$
$$f_1\leqslant f\leqslant f_2\leqslant f_A\leqslant f_B. \tag{65}$$

Similarly, cases (56) and (60) both require

$$f_A\leqslant f_B\leqslant f_3\leqslant f. \tag{66}$$

Multiple roots occur in the two excluded cases $C_0=0$ and $C_0=A_0$, but also in the allowed case

$$C_0=-2A_0. \tag{67}$$

For $A_0<0$ we obtain, from Eq. (55),

$$R_2=\frac{6|A_0|}{B_0}\frac{1}{\cosh^2[\sqrt{|A_0|}(\xi-\xi_0)]},$$
$$R_1=\frac{6|A_0|}{\sqrt{-B_0D_0}}\frac{\sinh[\sqrt{|A_0|}(\xi-\xi_0)]}{\cosh^2[\sqrt{|A_0|}(\xi-\xi_0)]}, \tag{68}$$

so that we must require $B_0D_0<0$.

Similarly, for $A_0<0$, Eq. (58) provides the solutions

$$R_2=-\frac{6|A_0|}{B_0}\frac{1}{\sinh^2[\sqrt{|A_0|}(\xi-\xi_0)]},$$
$$R_1=\frac{6|A_0|}{\sqrt{B_0D_0}}\frac{\cosh[\sqrt{|A_0|}(\xi-\xi_0)]}{\sinh^2[\sqrt{|A_0|}(\xi-\xi_0)]}, \tag{69}$$

with the requirement $B_0D_0>0$.

For $A_0>0$ it is solution (57) that is relevant, and yields

$$R_2=-\frac{6A_0}{B_0}\frac{1}{\sin^2[\sqrt{A_0}(\xi-\xi_0)]},$$
$$R_1=\frac{6A_0}{\sqrt{B_0D_0}}\frac{\cos[\sqrt{A_0}(\xi-\xi_0)]}{\sin^2[\sqrt{A_0}(\xi-\xi_0)]},\quad B_0D_0>0. \tag{70}$$

(2) Case (33): $C_0=0$
Equation (52) is

$$f'^2=f^3-2A_0f^2+\frac{4}{3}A_0^2f+\gamma_0\frac{4B_0^2}{9}, \tag{71}$$

with γ_0 arbitrary, and

$$R_2 = -\frac{3}{2B_0}f, \quad R_1 = \frac{1}{2\sqrt{B_0 D_0}}(-3f+2A_0), \quad B_0 D_0 > 0. \tag{72}$$

A multiple root, namely, a triple one, occurs in one case only

$$f'^2 = \left(f - \frac{2}{3}A_0\right)^3, \quad \gamma_0 = -\frac{2A_0^3}{3B_0^2}, \tag{73}$$

and we have

$$R_2 = -\frac{1}{B_0}\left(\frac{6}{(\xi-\xi_0)^2} + A_0\right),$$

$$R_1 = \frac{-6}{\sqrt{B_0 D_0}} \frac{1}{(\xi-\xi_0)^2}. \tag{74}$$

In all other cases, two of the roots f_i are complex, and f is given in Eqs. (60) and (61).

(3) Case (37): $C_0 = A_0$

Equation (52) in this case is

$$f'^2 = f^3 - A_0 f^2 + \gamma_0 \frac{4B_0^2}{9}, \tag{75}$$

where γ_0 is arbitrary, and we have

$$R_2 = -\frac{3}{2B_0}f, \quad R_1 = -\frac{3}{2\sqrt{B_0 D_0}}f, \quad B_0 D_0 > 0. \tag{76}$$

The discriminant of the cubic equation $f'^2 = 0$ is

$$D = \tfrac{16}{9}(A_0^3 - 3B_0^2\gamma_0)B_0^2\gamma_0. \tag{77}$$

For $D<0$, two roots are complex; for $D>0$, they are all real and distinct; and for $D=0$, we have a double or triple root. The absence of a first degree term in Eq. (75) implies that for f_i real the three signs of f_i cannot be all the same. Moreover, if a root is equal to $f_i = 0$, it must be double. The elementary solutions in this case are as follows.

(1) $\gamma_0 = A_0^3/3B_0^2$. The finite solitary wave, the singular solitary wave, the singular periodic solution, and the singular algebraic solutions in this case are

$$R_2 = \sqrt{\frac{D_0}{B_0}}R_1 = -\frac{A_0}{2B_0}\left[3\tanh^2\left(\frac{\sqrt{A_0}}{2}(\xi-\xi_0)\right) - 1\right], \quad A_0 > 0, \tag{78}$$

$$R_2 = \sqrt{\frac{D_0}{B_0}}R_1 = -\frac{A_0}{2B_0}\left[3\coth^2\left(\frac{\sqrt{A_0}}{2}(\xi-\xi_0)\right) - 1\right], \quad A_0 > 0, \tag{79}$$

$$R_2 = \sqrt{\frac{D_0}{B_0}}R_1 = -\frac{|A_0|}{2B_0}\left[\frac{3}{\sin^2\left(\frac{\sqrt{|A_0|}}{2}(\xi-\xi_0)\right)} - 2\right], \quad A_0 < 0, \tag{80}$$

$$R_2 = \sqrt{\frac{D_0}{B_0}}R_1 = -\frac{6}{B_0(\xi-\xi_0)^2}, \quad A_0 = 0, \tag{81}$$

respectively.

(2) $\gamma_0 = 0$. The four different real solutions in this case are

$$R_2 = \frac{3}{2B_0}|A_0|\,\mathrm{sech}^2\left(\frac{\sqrt{|A_0|}}{2}(\xi-\xi_0)\right), \quad A_0 < 0 \tag{82}$$

$$R_2 = -\frac{3}{2B_0}|A_0| \frac{1}{\sinh^2\left(\frac{\sqrt{|A_0|}}{2}(\xi-\xi_0)\right)}, \quad A_0 < 0 \tag{83}$$

$$R_2 = -\frac{3A_0}{2B_0} \frac{1}{\sin^2\left(\frac{\sqrt{A_0}}{2}(\xi-\xi_0)\right)}, \quad A_0 > 0 \tag{84}$$

and for $A_0 = 0$ we reobtain solution (81).

C. Solutions for $\alpha = 0$

Equation (52) [obtained from Eq. (49)] in this case is

$$f'^2 = f^3 - 4A_0 f^2 - 2B_0 S f - 4M^2 B_0^2, \tag{85}$$

where S and M are arbitrary. All possibilities for the three roots f_i occur, but we have the restriction

$$f_1 f_2 f_3 \geqslant 0. \tag{86}$$

We have

$$R_2 = -\frac{1}{2B_0}f, \quad R_1 = \sqrt{-\epsilon\frac{(2\kappa_1 + \beta + 2\delta\omega_1)}{2B_0}f}. \tag{87}$$

For R_1 to be globally defined, we need f to be sign definite. This imposes the following restrictions on the roots for each of the solutions of Sec. IV A.

Solutions (54) and (55):

$$f_1 \leqslant f \leqslant f_2 \leqslant 0 \leqslant f_3 \quad \text{or} \quad 0 \leqslant f_1 \leqslant f \leqslant f_2 \leqslant f_3. \tag{88}$$

Solutions (56)–(58):

$$0 \leqslant f_1 \leqslant f_2 \leqslant f_3 \leqslant f \quad \text{or} \quad f_1 \leqslant f_2 \leqslant 0 \leqslant f_3 \leqslant f. \tag{89}$$

Solution (59):

$$0 \leqslant f_1 = f_2 = f_3 \leqslant f. \tag{90}$$

Solution (60):

$$0 \leqslant f_3 \leqslant f. \tag{91}$$

V. OTHER EXPLICIT SOLUTIONS

In Sec. IV we presented solutions of Eq. (1) satisfying $a_1 a_2 \neq 0$, and such that $|a_1|$ and $|a_2|$ are not constant. Let us now discuss these previously rejected solutions.

(1) Solutions with $a_1 = 0$

For $\alpha \neq 0$, we put

$$a_2(x,t) = \omega(x,t) \exp i\left[-\left(\beta + \frac{\delta^2}{2\alpha}\right)t - \frac{\delta}{\alpha}x\right]. \tag{92}$$

Equation (1) for $a_2(x,t)$ and $a_1 = 0$ reduces to the linear Schrödinger equation

$$i\omega_t - \frac{\alpha}{2}\omega_{xx} = 0. \tag{93}$$

For $\alpha = 0$, the solution of Eq. (1) is

$$a_2(x,t) = e^{-i\beta t}\omega(\xi), \quad \xi = x + \delta t \tag{94}$$

where $\omega(\xi)$ is an arbitrary (complex) function.

(2) Solutions with R_1 and R_2 constant

We require that ϕ_1 and ϕ_2 should not be constant; otherwise we obtain $a_1 = a_2 = 0$. Equations (6)–(9) for R_1 and R_2 constant imply that we must have sin ϕ=0, i.e., the solutions will be phase locked.

Explicitly, for $\alpha \neq 0$ we have

$$R_2 = -\frac{r\omega^2}{2\epsilon}(\gamma_1^2 - 2A\gamma_1 + D),$$

$$R_1^2 = \frac{\alpha r \omega^4}{4}(4\gamma_1^2 - 2B\gamma_1 + C)(\gamma_1^2 - 2A\gamma_1 + D),$$

$$\phi_1 = \gamma_1 \xi + \gamma_2,$$

$$\phi_2 = 2\gamma_1 \xi + 2\gamma_2 + k\pi, \quad \epsilon = (-1)^k, \tag{95}$$

where γ_1 and γ_2 are arbitrary constants.

For $\alpha = 0$ we have ϕ_1 and ϕ_2, as in Eq. (95):

$$R_2 = -\epsilon\left[\frac{r\omega^2}{2}\gamma_1^2 + (v - r\omega\omega_1)\gamma_1 + \frac{r\omega_1^2 - 2\kappa_1}{2}\right],$$

$$R_1^2 = -\epsilon[(v + \delta\omega)\gamma_1 - (2\kappa_1 + \beta + \delta\omega_1)]R_2. \tag{96}$$

VI. CONCLUSION

The symmetries used in this paper are only those that exist in the generic case of the cascading equations (1), i.e., for all values of the constants r, α, β, and δ. This ''generic'' symmetry algebra is summed up in Eq. (3) representing space and time translations and a shift in the phases of the fundamental and second harmonic waves.

The reduction to the system of ordinary differential equations (6)–(9) was achieved by requiring that solutions be invariant under the one-dimensional subgroup of the symmetry group, corresponding to the Lie algebra element (4). In order to decouple these equations we had to impose either $a_1 = 0$ (no fundamental harmonic), or the supplementary condition (10) on the phases (phase locked solutions). The existence of the symmetry W in Eq. (3) guarantees that if the phases locking condition (10) is imposed on the initial conditions, it will survive for all times. Thus, by construction, all the explicit solutions obtained in this paper are phase locked traveling waves. Let us discuss some of their features.

Solution (63), for certain values of the parameters r, α, β, and δ, characterizing the material involved, and of the constants ω, v, ω_1, and κ_1, characterizing initial conditions, can be periodic finite waves [see Eq. (54)]. The second harmonic R_2 then oscillates in the interval $[-(3/2B_0)f_1, -(3/2B_0)f_2]$ and the fundamental wave also oscillates between finite limits.

For other conditions [see Eq. (67)], the elliptic function solutions reduce to solitary waves (68) with the second harmonic going through a zero when the fundamental one reaches its maximum value. Asymptotically both waves (68) tend to zero, with the second harmonic vanishing at a faster rate. Many of the obtained solutions are singular, either at some specific point $\xi = \xi_0$ as in Eq. (69), or periodically, as in Eqs. (56) or (60).

For conditions leading to Eq. (33), the second harmonic differs from the fundamental one just by a proportionality factor and a constant shift of the amplitude [see Eq. (35)]. Similarly, for conditions (37), the two waves are simply proportional.

Singular solutions coexist with the finite ones for all the values of the coefficients in Eq. (1). Their physical meaning needs a separate investigation: they may be an indication that higher harmonics or dissipative effects were unjustifiably ignored in the derivation of these equations. These effects would tend to smooth out the singularities and possibly turn them into finite resonance phenomena.

The existence of families of elliptic function solutions can be viewed as a manifestation of ''partial integrability.'' We have shown that the studied equations do not, in general, have the Painlevé property. For special values of the constants involved we do obtain solutions that do have this property: they have no movable singularities other than poles.

Finally let us mention that for particular values of the constants in Eq. (1), the symmetry group may be larger. For instance, for $\beta = \delta = 0$ the equations are invariant under dilatations generated by

$$D = x\partial_x + 2t\partial_t - 2(R_1\partial_{R_1} + R_2\partial_{R_2}).$$

This raises the possibility of obtaining self-similar solutions of the forms

$$a_1(\xi)=\frac{1}{t}F_1(\xi)e^{i\phi_1(\xi)}e^{(i/2)\ln t},$$

$$a_2(\xi)=\frac{1}{t}F_2(\xi)e^{i\phi_2(\xi)}e^{i\ln t},$$

$$\xi=\frac{x}{\sqrt{t}}.$$

Since self-similar solutions are particularly important (and stable) in optical systems with memory [20–22], this situation may be well worth exploring.

For elliptic solutions, which we have found to be observed experimentally, one would have to launch a series of parallel beams similar to the single beams described in Ref. [2]. At present, it does not appear to be feasible to do so. However, such a periodic array, should it ever become feasible, could be used as a reconfigurable, all-optical switching fabric. Thus it seems a long-term goal that is worth pursuing. Along these lines, we note that many years intervened between the first theoretical prediction of cascading solitons and their observation.

ACKNOWLEDGMENTS

The authors thank D. Levi, Z. Thomova, and Yu.S. Kivshar for helpful discussions. S.L. acknowledges financial support from NSERC (National Science and Engineering Research Council of Canada) for his Ph.D. studies. The research of P.W. was partly supported by research grants from NSERC of Canada and FCAR du Québec.

[1] See, e.g., G. P. Agrawal, *Nonlinear Fiber Optics* (Academic, San Diego, 1989), Chap. 2.

[2] W. E. Torruellas, Z. Wang, D. J. Hagan, E. W. Van Strykland, G. I. Stegeman, L. Torner, and C. R. Menyuk, Phys. Rev. Lett. **74**, 5036 (1995); R. Schiek, Y. Baek, and G. I. Stegeman, Phys. Rev. E **53**, 1138 (1996).

[3] C. R. Menyuk, R. Schiek, and L. Torner, J. Opt. Soc. Am. B **11**, 2434 (1994).

[4] L. Torner, C. R. Menyuk, W. E. Torruellas, and G. I. Stegeman, Opt. Lett. **20**, 13 (1995).

[5] A. V. Buryak and Yu. S. Kivshar, Opt. Lett. **19**, 1612 (1994).

[6] M. J. Werner and P. D. Drummond, J. Opt. Soc. Am. B **10**, 2390 (1993).

[7] M. J. Werner and P. D. Drummond, Opt. Lett. **19**, 613 (1994).

[8] P. Ferro and S. Trillo, Phys. Rev. E **51**, 4994 (1995).

[9] G. I. Stegeman, M. Sheik-Bhae, E. Van Strykland, and G. Assanto, Opt. Lett. **18**, 13 (1993).

[10] A. V. Buryak and Yu. S. Kivshar, Phys. Lett. A **197**, 407 (1995).

[11] A. V. Buryak and Yu. S. Kivshar, Phys. Rev. A **51**, R41 (1995).

[12] A. V. Buryak and Yu. S. Kivshar, Opt. Lett. **20**, 834 (1995).

[13] M. J. Ablowitz, A. Ramani, and H. Segur, J. Math. Phys. **21**, 715 (1980).

[14] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, (SIAM, Philadelphia, 1981).

[15] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, Berlin, 1986).

[16] B. Champagne, W. Hereman, and P. Winternitz, Comput. Phys. Commun. **66**, 319 (1991).

[17] P. Winternitz, in *Integrable Systems, Quantum Groups and Quantum Field Theory* (Kluwer, Dordrecht, 1993).

[18] D. Rand and P. Winternitz, Comput. Phys. Commun. **42**, 359 (1986).

[19] P. F. Byrd and M. D. Friedmann, *Handbook of Elliptic Integrals for Engineers and Scientists* (Springer, New York, 1971).

[20] D. Levi, C. R. Menyuk, and P. Winternitz, Phys. Rev. A **44**, 6057 (1991); **49**, 2844 (1994).

[21] C. R. Menyuk, D. Levi, and P. Winternitz, Phys. Rev. Lett. **69**, 3048 (1992).

[22] *Self-Similarity in Stimulated Raman Scattering*, edited by D. Levi, C. R. Menyuk, and P. Winternitz (CRM, Montréal, 1994).