Accurate Solution of the Paraxial Wave Equation Using Richardson Extrapolation

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Abstract—**Richardson extrapolation is used along with the mid-step Euler finite difference method to solve the paraxial wave equation. Highly accurate solutions can be efficiently obtained using this combined approach. Numerical results are presented for wave propagation in a straight integrated optical waveguide and a *Y*-junction waveguide. Discretization errors in both the propagation and the transverse dimension can be systematically eliminated by using Richardson extrapolation. Accuracies on the order of 10^{-11} or better are obtained.**

I. Introduction

ACCURATE analysis of guided wave devices such as waveguides and couplers is important for the development of optical integrated circuits. In general, numerical methods are used to investigate the propagation characteristics of such devices. The beam propagation method, which is based on the fast Fourier transform, has been used for a long time [1]; scalar and vector finite difference schemes have also been used [2]–[4]; more recently, a hybrid finite difference scheme has been implemented [5]. A straightforward way to improve accuracy is to increase the density of the grid and decrease the step size. With a uniform mesh and step size, the computational cost will quickly become prohibitive. Nonuniform mesh structures and step sizes [2] can achieve higher accuracies, but they are cumbersome to implement. It is useful to have highly accurate, yet rapid schemes for solving the paraxial wave equation in order to be able to separate inaccuracies inherent in the numerical methods from inaccuracies due to the paraxial wave approximation itself.

In this letter, we use a scheme based on Richardson extrapolation which allows one to efficiently and simply obtain a highly accurate solution of the paraxial wave equation. This extrapolation method is often used in numerical analysis [6], [7], but has not been applied to optical waveguide problems. Richardson extrapolation is used to eliminate discretization errors in both the propagation and the transverse directions.

In the following, we will discuss Richardson extrapolation and its implementation to solve the paraxial equation with one transverse dimension based on the explicit mid-step Euler finite difference method. We stress that Richardson extrapolation is a simple algebraic procedure which can be used in conjunction with any method to improve its accuracy. The explicit mid-step Euler method vectorizes well on a CRAY-YMP, the computer which we used, in contrast to implicit schemes which involve tridiagonal matrix inversions and do not vectorize well. Thus, the explicit mid-step Euler method is significantly more efficient. It is, however, unconditionally unstable, and consequently, has been rarely used in problems with one transverse dimension. However, it has long been known that Richardson extrapolation can stabilize unstable schemes [8], and we have found that it stabilizes the explicit mid-step Euler method in optical waveguide problems. As a consequence, the CPU cost of our approach is competitive with implicit schemes which have far lower accuracy.

As a particular example, we study the evolution of a Gaussian input pulse along a straight, single-mode, step-index waveguide. We use transparent boundary conditions [9] which are meant to be used in conjunction with the implicit Crank-Nicholson method. They also work well with our explicit mid-step Euler method when combined with Richardson extrapolation for reasons which will be explained in detail elsewhere. The purpose of this example is to examine the evolution of an initial pulse which is not perfectly matched to the waveguide and produces radiation—a case which often occurs in practice. No analytical results are available in this case; so, it is useful to have a highly accurate scheme available for use as a baseline. We also study a Y-junction optical waveguide in order to demonstrate that our approach works in a more complicated geometry.

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II. Computational Approach

In Richardson extrapolation, one uses a sequence of estimates which are obtained by varying the step size of a numerical calculation and extrapolating to zero step size. We may write

$$A(h,N) = S(h) + \epsilon_2(h)\frac{1}{N^2} + \epsilon_3(h)\frac{1}{N^3} + \cdots, \qquad (1)$$

where $A(h,N)$ is the outcome of the numerical calculation over a very short interval h, S is the exact solution, and $\Delta = h/N$ is the step size used inside the interval. More generally, one can have a term proportional to $1/N$ in the sequence. However, since the mid-step Euler method is second-order accurate and there are N steps inside h, the first error term will be proportional to $1/N^2$. From (1), the leading error terms can be eliminated by taking a linear combination of the results calculated by using different values of N. For example, we find

$$\frac{4A(h,2) - A(h,1)}{3} = S(h) - \frac{1}{6}\epsilon_3(h) + \cdots \qquad (2)$$

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After each stage of extrapolation, the accuracy of the solution can be estimated by comparing the new solution to the best solution obtained in the previous Richardson stage.

To illustrate the use of Richardson extrapolation in simulations of optical devices, we consider the paraxial wave equation,

$$2jk_0n_0\frac{\partial E}{\partial z} = \frac{\partial^2 E}{\partial x^2} + k_0^2[n^2(x,z) - n_0^2]E, \qquad (3)$$

where E is the electric field vector of a TE wave and k_0 is the wavenumber in free space. The parameter n_0 is the reference refractive index, and $n(x,z)$ is the cross section index profile. Eq. (3) is then solved numerically using the mid-step Euler finite difference method,

$$2jk_0n_0\frac{E_n^{m+\frac{1}{2}} - E_n^m}{\Delta z/2} = \frac{E_{n+1}^m - 2E_n^m + E_{n-1}^m}{\Delta x^2} + k_0^2\left[(n_n^m)^2 - n_0^2\right]E_n^m, \qquad (4a)$$

$$2jk_0n_0\frac{E_n^{m+1} - E_n^m}{\Delta z} = \frac{E_{n+1}^{m+\frac{1}{2}} - 2E_n^{m+\frac{1}{2}} + E_{n-1}^{m+\frac{1}{2}}}{\Delta x^2} + k_0^2\left[\left(n_n^{m+\frac{1}{2}}\right)^2 - n_0^2\right]E_n^{m+\frac{1}{2}}, \qquad (4b)$$

where $E_i^j = E(i\Delta x, j\Delta z)$, while Δx and Δz are step sizes in the transverse and propagation directions respectively.

In solving the paraxial equation, both the propagation and the transverse directions are discretized. To obtain an accurate solution, the discretization errors arising from both the transverse grid and the propagation step size have to be eliminated. Similar to (1), the solution can be written as

$$A(h_z, h_x, N, M) = S(h_z, h_x) + f_0(h_z, h_x, M) + f_1(h_z, h_x, M)\frac{1}{N} + f_2(h_z, h_x, M)\frac{1}{N^2} + \cdots \qquad (5)$$

where S is the exact solution over a very short interval of h_z in the propagation direction and a width of h_x in the transverse dimension, $\Delta z = h_z/N$ is the step size used inside h_z, and $\Delta x = h_x/M$ is the step size used inside h_x. The parameters M and N are integers, and

$$f_i(h_z, h_x, M) = \eta_{0,i}(h_z, h_x) + \eta_{2,i}(h_z, h_x)\frac{1}{M^2} + \eta_{4,i}(h_z, h_x)\frac{1}{M^4} + \cdots \qquad (6)$$

The coefficients $\eta_{0,0}$ and $\eta_{0,1}$ are equal to zero because the method is second order accurate. While the discretization error in the propagation direction can be eliminated by linear combinations of solutions with different N, and the error in the transverse dimension can be eliminated by using different transverse grid sizes.

III. Numerical Results

We calculated electric fields at each propagation step using (4). We then extrapolated the solutions until the error was below a given threshold. We used error thresholds of 10^{-8} and 10^{-12}. The error estimate is defined as $\int |e|^2 dx = \sum_{i=1}^{n} |E_i - E_i'|^2 \Delta x$ where the subscript i indicates the grid location in transverse dimension, and n is the number of grid points in the transverse dimension, E is the extrapolated solution, and E' is the solution obtained in the previous Richardson stage. The input pulse is normalized to unit power. The total propagation length is 4000 μm in the straight guide problem, while the total length is 500 μm in the Y-junction problem. The branching angle of the Y-junction is 3°. We carried out our calculations with 512, 1024, and 2048 transverse grid points. Then, the solutions obtained for different numbers of grid points were extrapolated in x to eliminate errors due to the discretization. The extrapolation in x need not be done at each step. In fact, for the problems we studied, we found that it was sufficient to extrapolate at the end of the calculations.

For both the straight waveguide and the Y-junction, the width of the guiding region was 4 μm and the computational window was 60 μm. The refractive indices of the guiding and the surrounding regions are 3.38 and 3.377 respectively. The reference refractive index n_0 is 3.377. The wavelength is 1.15 μm and the FWHM of the input Gaussian pulse is 4.828 μm. For the straight guide problem, we used propagation step sizes of 0.4, 0.067, and 0.013 μm for, respectively, 512, 1024, and 2048 transverse grid points when we set the error threshold at 10^{-8}, while we used step sizes of 0.2, 0.067, and 0.013 respectively when we set the error threshold at 10^{-12}. The computation time can be further improved by using variable step size in the z-evolution of the pulse [6] as we will discuss elsewhere.

In Fig. 1, we show the error estimate for each z-step for up to 3 stages of Richardson extrapolation for the wave evolution in the straight waveguide problem for a distance of 250 μm. In the case shown here, the error threshold is 10^{-12}, the basic step size is 0.25 μm, and there are 512 grid points. We use only as many stages as are required for the error to go below the threshold. A dramatic reduction in error is obtained with each additional iteration.

In Fig. 2, we plot the radiation intensity at distances of 1000, 2000, 3000, and 4000 μm along the straight waveguide. We use a semilog plot which shows the radiation leaving the guiding region. We have extrapolated these solutions for accuracy in the transverse direction. The estimated error after two stages of Richardson extrapolation in the transverse dimension is shown in Table I. In column I, we used an error threshold of 10^{-8} while propagating in the z direction, and in column II, we used an error threshold of 10^{-12}. Note that when the error threshold is set at 10^{-8}, the estimated accuracies are actually lower than the error threshold for distances less than 3000 μm. That happened because the actual error was significantly below threshold in this case as determined by comparison with the case in which the error threshold was 10^{-12}.

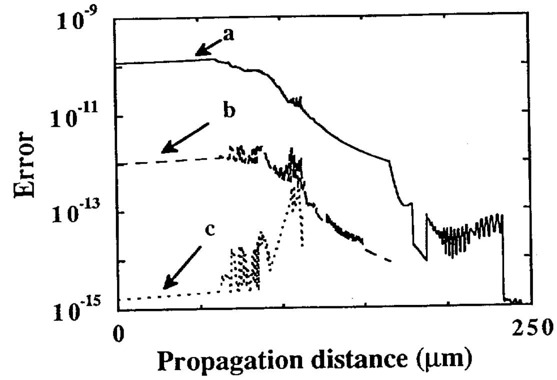

Fig. 1. Error in each propagation step vs. distance along the waveguide. The error is calculated with a grid of 512 points in x.(a) The error in each step after performing one stage of Richardson extrapolation, (b) The error after performing two stages of extrapolation, (c) The error after three stages of extrapolation. An error threshold of 10^{-12} is used.

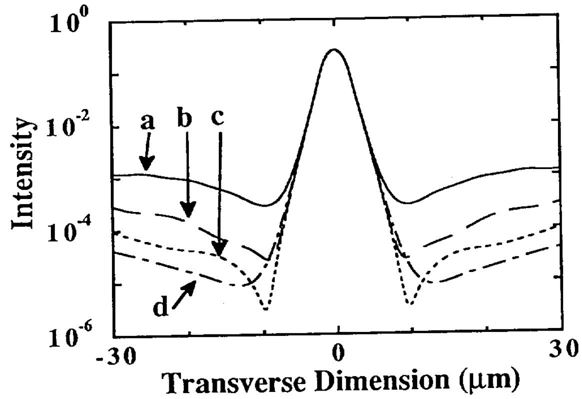

Fig. 2. Accurate solutions obtained after propagating for (a) 1000 μm, (b) 2000 μm, (c) 3000 μm, and (d) 4000 μm.

TABLE I
EsSTIMATED ERROR AFTER TWO STAGES OF RICHARDSON EXTRAPOLATION IN THE TRANSVERSE DIMENSION. ERROR THRESHOLDS OF 10^{-8} AND 10^{-12} ARE USED FOR THE z-PROPAGATION.

Distance	Error Thresholds	
μm	10^{-8}	10^{-12}
1000	1.7×10^{-11}	9.7×10^{-12}
2000	2.7×10^{-10}	1.6×10^{-11}
3000	6.8×10^{-09}	1.2×10^{-11}
4000	6.3×10^{-08}	1.1×10^{-11}

Finally, we compare the efficiency of the proposed scheme with the Crank-Nicholson implicit finite difference method. For the latter, we use 2048 transverse grid point and a propagation step size of 0.0285 μm. The simulation takes approximately 18 minutes of CRAY-YMP time and attains an accuracy on the order of 10^{-6}. Using the mid-step Euler method and Richardson extrapolation, only 14 minutes is required to attain an accuracy on the order of 10^{-11}. For a Y-junction waveguide, while using mid-step Euler method with Richardson extrapolation 2 minutes of Cray time is required to obtain an accuracy of 10^{-8}, while the Crank-Nicholson method required 4 minutes to obtain an accuracy of 10^{-6}. These results demonstrate that Richardson extrapolation is an efficient way to obtain accurate solutions to optical waveguide problems.

IV. CONCLUSION

In this letter, we present a simple and efficient approach, based on a combination of the mid-step Euler method with Richardson extrapolation, to obtain an accurate solution of the paraxial wave equation. We use the extrapolation scheme to reduce discretization errors in both the transverse and propagation dimensions. We study a straight waveguide and a Y-junction waveguide, and we obtain solutions with accuracies on the order of 10^{-8} to 10^{-12}. Comparison with the usual Crank-Nicholson method shows that this approach is efficient and highly accurate.

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