Stability of black solitons in media with arbitrary nonlinearity

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It is shown that the black solitons in an optical fiber or a uniform medium with arbitrary nonlinearity are all stable. The conclusion from the analytical stability analysis is consistent with that of numerical simulations. This then dismisses a previous criterion that suggests that the black solitons in a saturable nonlinear medium can be unstable.

The propagation of solitons in an optical fiber or in a uniform medium has been a subject of great interest since the early 1970's[1,2] because of their potential applications in long-distance communication and information processing. In a Kerr-law nonlinear medium the soliton states are stable, and they survive even through collision, as demonstrated in the earlier research of Zakharov and Shabat.[1] In reality, nonlinearity deviates from the ideal Kerr-law case. For example, non-Kerr-law nonlinearity can arise from nonlinear saturation or from doping silica fibers with several different materials. In a saturable nonlinear medium or a non-Kerr-law medium, soliton trapping can differ qualitatively or quantitatively from its Kerr-law counterpart,[3–10] and so can the corresponding stability. For the fundamental bright solitons the stability of the trapped states in an arbitrary nonlinear medium was shown to be related to the sign of $\mathrm{d}P/\mathrm{d}\beta$, where P is the soliton energy or power and β is the propagation constant.[11–13] $\mathrm{d}P/\mathrm{d}\beta > 0$ corresponds to stable stationary solutions, and $\mathrm{d}P/\mathrm{d}\beta < 0$ to unstable stationary solutions.[11–13] For dark solitons in an optical fiber operating within the normal dispersion regime or in a self-defocusing uniform medium there appears to be some confusion concerning the stability for non-Kerr law nonlinearity. Based on the Kerr-step-Kerr nonlinearity model, Enns and Mulder[6] suggested that, for the dark solitons, $\mathrm{d}P_c/\mathrm{d}A > 0$ is associated with stable dark-soliton solutions and $\mathrm{d}P_c/\mathrm{d}A < 0$ with unstable dark-soliton solutions, where P_c is the complementary power[6] and A is the contrast of dark solitons, with $A = 1$ referring to black solitons. Krolikowski *et al.*[14] later showed that the criterion of $\mathrm{d}P_c/\mathrm{d}A$ is not sufficiently general and does not work for the saturable nonlinear medium that they considered. Instead, they proposed that $\mathrm{d}Q/\mathrm{d}V < 0$ (>0) is a criterion related to stable (unstable) dark-soliton solutions, where Q is the momentum and V is a parameter that measures the steering angle of the dark solitons, with $V = 0$ corresponding to black solitons.[15] They then concluded that the dark solitons with V smaller than a critical value V_{cr}, below which $\mathrm{d}Q/\mathrm{d}V > 0$, are all unstable.[15] In other words, their criterion involving the momentum Q indicates that black solitons ($V = 0$ or $A = 1$) can be unstable in saturable nonlinear media.[15] We find that this conjecture[15] involving the momentum Q is not valid for black solitons because black solitons (odd functions with zero intensity at the center), as we show below (consistent with an earlier report[11]), are all stable for an arbitrary nonlinearity.

The evolution of a soliton pulse in an optical fiber operating within the normal dispersion regime or a soliton wave in a uniform self-defocusing medium is governed by the nonlinear Schrödinger equation. In normalized units, the nonlinear Schrödinger equation can be written as

$$i\frac{\partial e}{\partial \xi} - \frac{1}{2}\frac{\partial^2 e}{\partial \tau^2} + f(|e|^2)e = 0\,, \tag{1}$$

where, for example in the case of a pulse, e is the normalized envelope function and τ and ξ are the normalized time and distance, respectively. $f(y)$ is an arbitrary function depicting the nonlinearity of the medium; $f(y) = y$ for Kerr-law nonlinearity, $f(y) = [1 - \exp(-\alpha_1 y)]/\alpha_1$ for exponential saturable nonlinearity, and $f(y) = y/(1 + y/I_s)$ for two-level saturable nonlinearity. The solutions to Eq. (1) for dark solitons have in general the form of $e(\tau, \xi) = q(T = \tau - V\xi, \xi)\exp(i\beta\xi)$, which, substituted into Eq. (1), leads to

$$i\frac{\partial q}{\partial \xi} - iV\frac{\partial q}{\partial T} - \frac{1}{2}\frac{\partial^2 q}{\partial T^2} + [f(|q|^2) - f(|q_\infty|^2)]q = 0\,, \tag{2}$$

where $\beta = f(|q_\infty|^2)$ is determined by the boundary condition of $|q| = |q_\infty|$ or $|e| = |e_\infty|$ at $T = \tau - V\xi = \pm\infty$.

Equation (2) or (1) is a Hamiltonian system and has two invariants, i.e., Hamiltonian and momentum[16]:

$$H = \int_{-\infty}^{\infty}\left[\frac{i}{2}V\left(q\frac{\partial q^*}{\partial T} - q^*\frac{\partial q}{\partial T}\right) + \frac{1}{2}\left|\frac{\partial q}{\partial T}\right|^2 + F(|q|^2)\right]\mathrm{d}T\,, \tag{3a}$$

$$Q = \frac{i}{2}\int_{-\infty}^{\infty}\left(q\frac{\partial q^*}{\partial T} - q^*\frac{\partial q}{\partial T}\right)\mathrm{d}T - |q_\infty|^2\arg(q)|_{-\infty}^{\infty}\,, \tag{3b}$$

with $F(|q|^2) - \int_{|q_\infty|^2}^{|q|^2}[f(y) - f(|q_\infty|^2)]\mathrm{d}y$. The stationary solutions of Eq. (2) are recovered from

$$\delta H = 0\,.$$

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This means that the stationary solutions of Eq. (2) are those functions $q = q_s$ at which Hamiltonian H achieves its maximum or minimum. $\delta^2 H > 0$ indicates that H at $q = q_s$ is a minimum and $q = q_s$ is a stable solution, whereas $\delta^2 H < 0$ implies that H is a maximum and $q = q_s$ is unstable. Thus the stability of the stationary solutions can be revealed by examination of the definiteness of $\delta^2 H$. By substituting $q = q_s + \delta q$ into Eq. (1) one can derive the second variation of the Hamiltonian $\delta^2 H$ and express it in terms of $q_s = u_s + iv_s$ and $\delta q = u + iv$:

$$\delta^2 H = \langle u|L_1 u\rangle + \langle v|L_0 v\rangle + \langle u|L_{01} v\rangle + \langle v|L_{10} u\rangle\,, \quad (4)$$

where $\langle f_1|f_2\rangle = \int_{-\infty}^{\infty} f_1 f_2 \mathrm{d}T$, $L_1 = -0.5\partial^2/\partial T^2 + f(|q_s|^2) - f(|q_\infty|^2) + 2{u_s}^2 f'(|q_s|^2)$, $L_0 = -0.5\partial^2/\partial T^2 + f(|q_s|^2) - f(|q_\infty|^2) + 2{v_s}^2 f'(|q_s|^2)$, $L_{01} = 2u_s v_s f'(|q_s|^2) + V\partial/\partial T$, $L_{10} = 2u_s v_s f'(|q_s|^2) - V\partial/\partial T$, and a prime indicates a derivative with respect to the argument. From the linearized equations

$$\partial v/\partial \xi = L_1 u + L_{01} v, \qquad -\partial u/\partial \xi = L_0 v + L_{10} u \quad (5)$$

obtained by substitution of $q = q_s + \delta q$ into Eq. (2), it is straightforward to show that in general the derivative of the stationary dark-soliton solutions is related to the perturbation function by

$$\langle \partial u_s/\partial T|v\rangle = \langle \partial v_s/\partial T|u\rangle\,. \quad (6)$$

For the special case of the black solitons of $V = 0$ and $v_s = 0$, Eq. (6) reduces to $\langle \partial u_s/\partial T|v\rangle = 0$. This means that the perturbation function v is orthogonal to the derivative of the stationary black-soliton solution $\partial u_s/\partial T$. This $\partial u_s/\partial T$ is in fact the fundamental state of the operator L_1 for the black solitons, i.e., $L_1 \partial u_s/\partial T = 0$. Therefore L_1 is positive definite, and the first term on the right-hand side of Eq. (4), $\langle u|L_1 u\rangle$, is always greater than or equal to zero. For black solitons the last two terms of Eq. (4) disappear, because $L_{01} = L_{10} = 0$. One can evaluate the minimum value of the second term on the right-hand-side of Eq (4), $\langle v|L_0 v\rangle$, by considering the following eigenvalue problem:

$$L_0 v = \lambda v + \alpha \partial u_s/\partial T \quad (7)$$

for minimum λ, where the constants α and λ are determined by the conditions of orthogonality, $\langle \partial u_s/\partial T|v\rangle = 0$, and normalization, $\langle v|v\rangle = 1$. This is so because estimation of the minimum value of $\langle v|L_0 v\rangle$ is equivalent to finding the smallest eigenvalue λ in Eq. (7).[17] If the minimum $\lambda > 0$, the second term on the right-hand side of Eq. (4) is positive and $\delta^2 H$ is positive definite. The black-soliton solution is stable. Otherwise it may be unstable.

Expanding $v(=\sum_{m=1}^{\infty} a_m v_m)$ and $\partial u_s/\partial T$ $(=\sum_{m=1}^{\infty} c_m v_m)$ of Eq. (7) in the complete set of eigenfunctions v_m of the operator L_0 gives rise to $v = \alpha \sum_{m=1}^{\infty} c_m v_m/(\lambda_m - \lambda)$. This expansion v, substituted into the orthogonality condition $\langle \partial u_s/\partial T|v\rangle = 0$, leads to an equation for determining λ:

$$g(\lambda) = \sum_{m=1}^{\infty} {c_m}^2/(\lambda_m - \lambda) = 0\,. \quad (8)$$

For the black solitons of $V = 0$ and $v_s = 0$, operator L_0 admits of only one negative eigenvalue λ_1 and a second eigenvalue $\lambda_2 = 0$ with eigenfunction $v_2 = u_s$, which is orthogonal to $\partial u_s/\partial T$, i.e., $c_2 = \langle v_2|\partial u_s/\partial T\rangle = 0$, because $L_0 u_s = 0$. Equation (8) then indicates that the smallest $\lambda_{\min} = \min(\lambda)$ must lie between λ_1 and the smallest positive eigenvalue of L_0. Also from Eq. (8), we have $\lambda = \lambda_{\min} > 0$ when $g(0) < 0$ and $\lambda = \lambda_{\min} < 0$ when $g(0) > 0$. $g(0)$ is related to the complementary power $P_c = \int_{-\infty}^{\infty}(|q_\infty|^2 - |q_s|^2)\mathrm{d}T$ by

$$g(0) = \sum_{m=1}^{\infty} {c_m}^2/\lambda_m = \langle \partial u_s/\partial T|{L_0}^{-1}\partial u_s/\partial T\rangle = -\langle \partial u_s/\partial T|u_s T\rangle = -0.5P_c\,, \quad (9)$$

where we have used relationships $\partial u_s/\partial T = \sum_{m=1}^{\infty} c_m v_m$ and $L_0(u_s T) = -\partial u_s/\partial T$. $g(0) = -0.5P_c < 0$ of Eq. (9) means that $\lambda = \lambda_{\min} > 0$. This indicates that $\min\langle v|L_0 v\rangle$ is a positive quantity, and therefore $\delta^2 H > 0$. We then conclude that the black solitons in a uniform medium with arbitrary nonlinearity are stable.

To confirm the stability result of the black solitons from analytical stability analysis, we conduct numerical simulations. In our numerical experiments we examine saturable nonlinear media, namely, exponential saturation, a two-level saturable model, and nonlinear saturation of $f(y) = 0.5I_s[1 - 1/(1 + y/I_s)^2]$. Numerical simulations show that the black solitons in saturable nonlinear media are stable for any saturation value. In particular, we propagate the black solitons in the saturable nonlinear model $f(y) = 0.5I_s[1 - 1/(1 + y/I_s)^2]$ considered in Ref. 15 for $I_s = 0.08$ and $I_s = 0.05$ by adding perturbations to the initial excitation of the stationary solutions. Small perturbations initially implanted radiate with propagation, leading to the stable steady-state evolution. The results from the analytical and numerical stability analyses demonstrated here are contrary to the conjecture of Ref. 15, which suggested that black solitons at the saturation values $I_s = 0.08$ and $I_s = 0.05$ are unstable (see Figs. 1 and 2 of Ref. 15).

Here, as illustrated, there appears to be a discrepancy between the stable black solitons in a uniform medium with arbitrary nonlinearity discussed above and earlier[11] and the unstable black solitons in some saturable nonlinear media as reported in Ref. 15. A question that immediately follows is: Where is the source of the discrepancy? The origin of the discrepancy lies in an invalid assumption made implicitly in Ref. 15 during a linear stability analysis. By an asymptotic expansion method[18] as used in Ref. 15, i.e., by replacing u, $v \sim \exp(\mu\xi)$ in the linearized equation (5) with $u = u_0 + \mu u_1 + \mu^2 u_2 + \mu^3 u_3 + O(\mu^4)$ and $v = v_0 + \mu v_1 + \mu^2 v_2 + \mu^3 v_3 + O(\mu^4)$ and substituting the subsequent equations into the orthogonality condition (6), we can show that the growth rate μ for a dark stationary solution is decided by

$$\mu^2 = \frac{\langle \mathbf{E}_t | \mathbf{L}^{-1}\mathbf{E}\rangle}{\langle \mathbf{W}_t | \mathbf{L}^{-1}\mathbf{W}\rangle} = \frac{-0.5\mathrm{d}Q/\mathrm{d}V}{\langle \mathbf{W}_t | \mathbf{L}^{-1}\mathbf{W}\rangle}, \tag{10}$$

where $\mathbf{L}^{-1}$ is the inverse operator of

$$\mathbf{L} = \begin{bmatrix} L_1 & L_{01} \\ L_{10} & L_0 \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} \partial v_s/\partial T \\ -\partial u_s/\partial T \end{bmatrix},$$

$$\mathbf{W} = \begin{bmatrix} v_1 \\ -u_1 \end{bmatrix} = \begin{bmatrix} -\partial v_s/\partial V \\ \partial u_s/\partial V \end{bmatrix},$$

and the subscript t refers to transpose. Obviously, if $\mu^2 > 0$, the dark stationary solution $q_s = u_s + iv_s$ is unstable. Otherwise, it is stable. In the analysis of Ref. 15 it is assumed that the denominator $D = \langle \mathbf{W}_t | \mathbf{L}^{-1}\mathbf{W}\rangle$ of Eq. (10) does not change sign as the soliton velocity V varies and that D is a negative quantity. This then yields $\mu^2 \sim \mathrm{d}Q/\mathrm{d}V$, leading to $\mathrm{d}Q/\mathrm{d}V > 0$ ($\mu^2 > 0$) associated with unstable dark stationary solutions for $V < V_{\mathrm{cr}}$ and $\mathrm{d}Q/\mathrm{d}V < 0$ ($\mu^2 < 0$) with stable stationary solutions for $V > V_{\mathrm{cr}}$ (see Fig. 2 of Ref. 15). The truth of the matter is that the sign of denominator D in Eq. (10) can vary with V, and the sign of $\mathrm{d}Q/\mathrm{d}V$ alone is not sufficient to determine the stability of a dark soliton. This becomes clear when one examines the case $V = 0$ of the black solitons, for which Eq. (10) reduces to

$$\mu^2 = \frac{\langle \partial u_s/\partial T | {L_0}^{-1}\partial u_s/\partial T\rangle}{\langle {L_0}^{-1}\partial u_s/\partial T | {L_1}^{-1}{L_0}^{-1}\partial u_s/\partial T\rangle}. \tag{11}$$

As shown above, L_1 is positive definite, and so is its inverse operator ${L_1}^{-1}$. Therefore the denominator $D = \langle {L_0}^{-1}\partial u_s/\partial T | {L_1}^{-1}{L_0}^{-1}\partial u_s/\partial T\rangle$ becomes positive as $V \to 0$ for the black solitons. Also from Eq. (9), the numerator of Eq. (11) is $\langle \partial u_s/\partial T | {L_0}^{-1}\partial u_s/\partial T\rangle = -0.5P_c < 0$. We then arrive at the same conclusion as before by considering the Hamiltonian of the system; i.e., the black solitons for arbitrary nonlinearity are stable even though $\mathrm{d}Q/\mathrm{d}V > 0$ as $\mu^2 < 0$.

Finally, it should be mentioned that physically stable black solitons in an optical fiber operating within the normal dispersion regime or in a self-defocusing uniform medium with arbitrary nonlinearity apparently arise from the fact that the stationary black-soliton solution sits just at a minimum of the Hamiltonian of the system, as illustrated in the above rigorous mathematical approach. A perturbed black stationary solution that deviates slightly from its minimum tends to be pushed back to the minimum point.

In conclusion, it has been shown that the black solitons in an optical fiber or in a uniform medium with arbitrary nonlinearity are all stable. The conclusion from the analytical stability analysis is consistent with an earlier report[11] and with that of numerical simulations. This then dismisses a previous criterion that suggested that black solitons in a saturable nonlinear medium can be unstable.

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References

1. V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP **34,** 62 (1972); **37,** 823 (1973).
2. A. Hasegawa and F. Tappert, Appl. Phys. Lett. **23,** 142, 171 (1973).
3. A. E. Kaplan, Phys. Rev. Lett. **55,** 1291 (1985); IEEE J. Quantum Electron. **QE-21,** 1538 (1985).
4. R. H. Enns, S. S. Rangnekar, and A. E. Kaplan, Phys. Rev. A **35,** 466 (1987); **36,** 1270 (1987).
5. S. Gatz and J. Herrmann, J. Opt. Soc. Am. B **8,** 2296 (1991).
6. R. H. Enns and L. J. Mulder, Opt. Lett. **14,** 509 (1989); IEEE J. Quantum Electron. **25,** 2205 (1989).
7. V. E. Wood, E. D. Evans, and R. P. Kenan, Opt. Commun. **69,** 156 (1988).
8. Y. Chen, Opt. Lett. **16,** 4 (1991).
9. Y. Chen, Phys. Rev. A **45,** 5215 (1992).
10. W. Krolikowski and B. Luther-Davies, Opt. Lett. **17,** 1414 (1992); **18,** 188 (1993) .
11. D. J. Mitchell and A. W. Snyder, J. Opt. Soc. Am. **10,** 1572 (1993).
12. N. G. Vakhitiov and A. A. Kolokolov, Sov. Radio Phys. **16,** 783 (1973).
13. C. K. R. T. Jones and J. V. Moloney, Phys. Lett. A **117,** 175 (1986).
14. W. Krolikowski, N. Akhmediev, and B. Luther-Davies, Phys. Rev. E **48,** 3980 (1993).
15. Y. S. Kivshar and W. Krolikowski, Opt. Lett. **20,** 1527 (1995).
16. Y. Chen, Phys. Rev. A **50,** 5205 (1994).
17. J. Atai, Y. Chen, and J. M. Soto-Crespo, Phys. Rev. A **49,** R3170 (1994).
18. V. E. Zakharov and A. M. Rubenchik, Sov. Phys. JETP **38,** 494 (1974).