Effect of third-order dispersion on dark solitons

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Third-order dispersion has a detrimental effect on dark solitons, leading to resonant generation of growing soliton tails and soliton decay. This effect is shown to be much stronger than that for bright solitons.

The recent experimental demonstration of data transmission based on dark solitons[1] has led to renewed efforts to explore this type of soliton for optical communications. Although the generation of a modulated stream of dark solitons is a far more difficult task than is the case for bright ones, dark solitons are usually believed to display some advantages over bright solitons, including lower Gordon–Haus jitter.[2] At the same time, one of the main problems associated with dark-soliton signal transmission is a relatively high average power. To reduce the power one should select the operational wavelength closer to the zero of the group-velocity dispersion, where, however, soliton propagation is influenced by third-order dispersion. The effect of third-order dispersion on bright solitons is now well understood.[3–5] Under the action of third-order dispersion a bright soliton develops a nonvanishing asymptote in the form of a tail.[3] Wai *et al.*[4] showed that the tail's amplitude A is exponentially small in the third-order dispersion coefficient β, $A \sim \exp(-1/\beta)$, and can be calculated by asymptotic analysis "beyond all orders." The rate of the energy emission is exceedingly small, e.g., of the order of 10^{-11} over one dispersion length at $\beta = 0.08$.[4] This result is consistent with Menyuk's robustness hypothesis,[6] according to which autonomous, homogeneous, Hamiltonian deformations of integrable equations lead to solitons that radiate beyond all orders if they radiate at all.

The effect of third-order dispersion on dark solitons has not been completely understood, although it has already been shown that third-order dispersion does not lead to tail generation for dark solitons of small amplitudes[7] (see also Ref. 8). Here we reveal that third-order dispersion leads to a destructive effect, including tail generation and soliton decay, for dark solitons of large and moderate amplitudes.

Nonlinear pulse propagation in optical fibers near the zero of the group-velocity dispersion is described by the nonlinear Schrödinger equation, modified to include the third-order dispersion,

$$i\frac{\partial u}{\partial z} - \alpha\frac{\partial^2 u}{\partial t^2} + 2|u|^2 u = i\beta\frac{\partial^3 u}{\partial t^3}, \quad (1)$$

where α stands for the dispersion type. Here we consider positive α only. As is well known, the case $\alpha > 0$ corresponds to the absence of modulational instability, and it can easily be verified that third-order dispersion does not modify this condition. For $\beta = 0$, Eq. (1) is exactly integrable, and its localized solutions are dark solitons propagating on a modulationally stable background of the intensity u_0, $u_s(x, t) = u_0(\cos\theta \tanh \Xi + i \sin\theta)\exp(2iu_0{}^2 z)$, where $\Xi \equiv u_0 \cos\theta(t - 2\lambda\sqrt{\alpha}u_0 z)/\sqrt{\alpha}$. Physical parameters such as the amplitude, the velocity, and the phase shift across the soliton are determined in terms of the phase angle θ, $|\theta| < \pi/2$.

In the case $\beta \neq 0$, dark solitons of Eq. (1) are not known and, in principle, localized solutions may not exist. Important information can be extracted from the analysis of the soliton asymptotics. We take $u = (u_0 + v)\exp(2iu_0{}^2 z)$ and linearize Eq. (1) for small v:

$$i\frac{\partial v}{\partial z} - \alpha\frac{\partial^2 v}{\partial t^2} + 2u_0{}^2(v + v^*) = i\beta\frac{\partial^3 v}{\partial t^3}\cdot$$

Then, for a stationary wave moving at velocity V, we seek a solution in the form $v \sim (v_r + iv_i)\exp(ik\zeta)$, where $\zeta = t - V_z$, and hence obtain an equation for $\kappa \equiv k^2(V)$:

$$(V - \beta\kappa)^2 = \alpha(\alpha\kappa + 4u_0{}^2). \quad (2)$$

Quadratic equation (2) has two roots. To investigate them we restrict our consideration to the soliton velocities $V^2 < c^2$, where $|c| = 2\sqrt{\alpha}u_0$ is the speed of sound. Then one root, κ_-, is always negative, and it corresponds to exponentially decaying soliton tails. However, for any $\beta \neq 0$ Eq. (2) has an additional, positive, root, κ_+, which describes a nonvanishing oscillatory tail of the soliton.

As in the case of bright solitons,[3,4] the existence of nonvanishing asymptotic behavior can be usefully viewed as a resonant generation of linear waves, which takes place provided that the speed of the solitary wave V coincides with the phase velocity V_{ph} of linear waves. Indeed, the condition $V_{\mathrm{ph}} = V$ leads immediately to Eq. (2). From the physical point of view, this result implies that the solitary wave acts as a source generating trailing oscillations; the oscillations propagate with the group velocity V_g. This process is demonstrated in Fig. 1 for the case of a black soliton of Eq. (1) with $\beta = 0.18$. At the initial stage of the soliton evolution some transiting radiation is excited. This radiation propagates to the left, moving with

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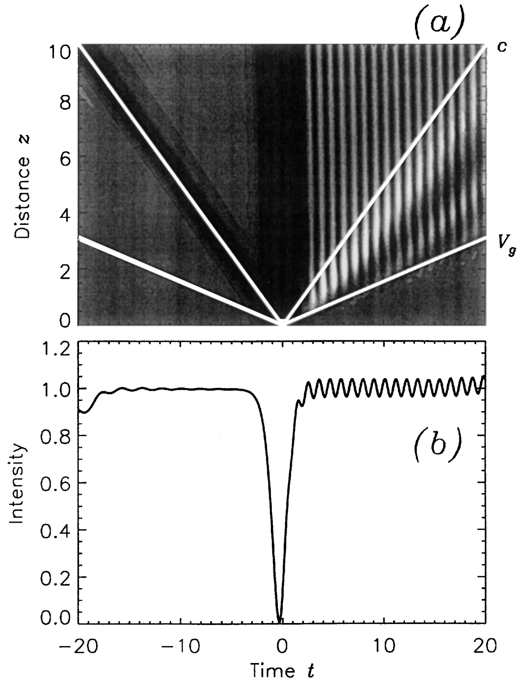

Fig. 1. Formation of a nonvanishing oscillating tail for the black soliton ($\alpha = u_0 = 1$) at $\beta = 0.18$. (a) Gray-scale plot in which the white lines give the propagation with velocity of sound c and the group velocity V_g. (b) Intensity profile at $z = 10$.

the speed of sound c, and quickly separates from the dark soliton, as shown in Fig. 1(a). The radiation creates an additional, small-amplitude dark soliton [see Fig. 1(b)]. An oscillating tail is formed from the right of the primary soliton, and its front propagates with group velocity V_g, which is different from the velocity of the dark soliton V and the velocity of sound c. Note that the oscillation amplitude and period are very stable, and the relative phase between the tail and the soliton remains constant. As a result of the generation of a continuously growing tail the soliton amplitude decreases and its velocity increases; i.e., the soliton decays (see Fig. 2). In Fig. 2(a) we plot a change of intensity at the soliton center, $I_{\min}$, versus propagation distance z, which varies from zero to approximately 0.8. Figure 2(b) proves that this change is almost adiabatic and can be described by the dependence $I_{\min} = {u_0}^2V^2/c^2$.

We also carried out calculations for gray solitons ($\theta \neq 0$). In this case we are able to distinguish clearly two stages of soliton dynamics. The first stage is a transition to a new, quasi-stationary state with a small change in the soliton's amplitude and velocity. This change is directly proportional to β, and the soliton's amplitude increases for $\beta\theta < 0$ and decreases otherwise. The first stage is also accompanied by the emission of radiation that either forms additional solitons or disperses. The second stage of the soliton's evolution is characterized by the generation of an oscillating tail. As the tail's amplitude depends exponentially on β [see Eq. (5) below], the first stage dominates for small β ($\beta < 0.1$) and the second stage dominates for relatively large β.

Calculation of the oscillation amplitude is a delicate task that requires all orders of the asymptotic expansion.[4,9] However, we can obtain a qualitatively correct result by considering the linear equation for the soliton perturbation, $\xi = u - u_s$:

$$i\frac{\partial \xi}{\partial z} - \alpha\frac{\partial^2 \xi}{\partial t^2} - i\beta\frac{\partial^3 \xi}{\partial t^3} + 2|u_s|^2\xi + {u_s}^2\xi^* = i\beta\frac{\partial^3(u_s)}{\partial t^3}\,. \tag{3}$$

A solution of Eq. (3) can be found in a cumbersome form, but its general structure is given by $\xi = A\Theta(\zeta)\Theta(-\zeta + V_g z)\sin(\sqrt{\kappa_+}\zeta + \phi)$, where $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$ and $\zeta = t - Vz$. Here we are interested primarily in the dependence of the tail amplitude A on β and V, for which we obtain

$$A \sim CB(\kappa_+)\mathrm{csch}\left(\frac{\pi\sqrt{\alpha}}{u_0 \cos\theta}\sqrt{\kappa_+}\right), \tag{4}$$

where C is a constant whose explicit value one can determine by taking into account all orders (cf., e.g., Ref. 10); the effects of all such terms are better found numerically by introduction of the yet unknown C. Function B is a rational function of the positive root of Eq. (2), κ_+. Using the first-order expansion for $\beta \to 0$, $k = \sqrt{\kappa_+} \approx \alpha/\beta$, one can easily show that the tail amplitude depends on β exponentially for fixed V (cf. Ref. 9):

$$A \sim \exp\left[-\frac{\pi\alpha^{3/2}}{u_0\beta(1 - V^2/c^2)^{1/2}}\right], \tag{5}$$

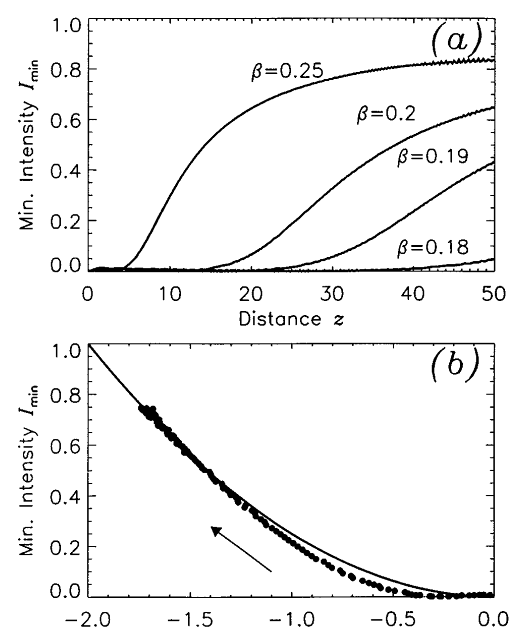

Fig. 2. Long-term adiabatic decay of a black soliton in the presence of third-order dispersion. (a) Intensity at the soliton center $I_{\min}(z)$, (b) minimum soliton intensity $I_{\min}$ for $\beta = 0.25$, calculated analytically as $I_{\min} = {u_0}^2V^2/c^2$ (solid curve) and from numerical simulations at equal distances (filled circles). The arrow shows the direction of the evolution.

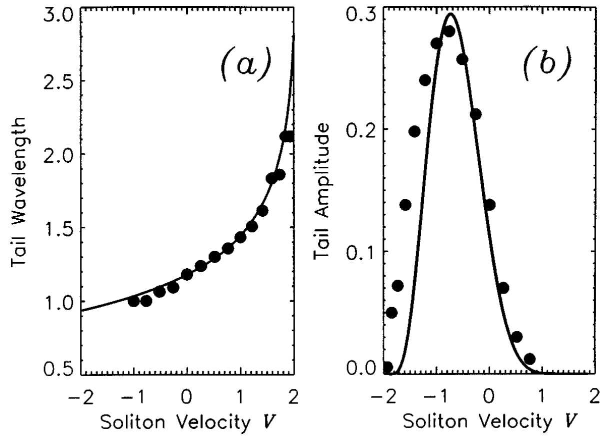

Fig. 3. Comparison of numerical (filled circles) and analytical (solid curves) results for (a) the wavelength and (b) the amplitude of the oscillating tail. Parameters are $\alpha = u_0 = 1$ and $\beta = 0.2$. The analytical curve in (a) is given by the resonant wavelength $\sqrt{\kappa_+}$; that in (b) is given by the expression for A from relation (4) at $c = 7000$.

i.e., in a fashion similar to that for bright solitons and in a manner that is consistent with the robustness hypothesis. Moreover, the algebraic factor $(1 - V^2/c^2)^{1/2}$ in the exponent demonstrates that the radiation amplitude becomes exponentially small for any fixed β in the limit $V^2 \rightarrow c^2$.

In general, the dependence on the soliton velocity V is less trivial than the dependence on β. Tail generation dominates for $\beta V < 0$. In particular, when $\beta = 0.2$ the tail amplitude reaches its maximum value at $V = -0.8$ (see Fig. 3). This result is in agreement with numerical simulations.

Now we can readily understand the validity of the small-amplitude approximation used in Ref. 7. Indeed, as the amplitude of the dark soliton decreases, i.e., as the soliton velocity V approaches the limit velocity c at a fixed value of β, the amplitude of the oscillating tail decreases rapidly because it depends exponentially on the soliton amplitude [see Eq. (5)]. The oscillation amplitude is beyond all orders of the asymptotic expansion in the soliton amplitude, and in this limit the soliton decay is therefore negligible.

In conclusion, we have demonstrated the destructive action of the third-order dispersion on dark solitons that results in the generation of growing tails and subsequent soliton decay.

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