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Integrability and Self-Similarity in Transient Stimulated Raman Scattering

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Summary. The phenomenon of stimulated Raman scattering (SRS) can be described by three coupled PDEs which define the pump electric field, the Stokes electric field, and the material excitation as functions of distance and time. In the transient limit these equations are integrable, i.e., they admit a Lax pair formulation. Here we study this transient limit. The relevant physical problem can be formulated as an initial-boundary value (IBV) problem where both independent variables are on a *finite* domain. A general method for solving IBV problems for integrable equations has been introduced recently. Using this method we show that the solution of the equations describing the transient SRS can be obtained by solving a certain linear integral equation. It is interesting that this equation is identical to the linear integral equation characterizing the solution of an IBV problem of the sine-Gordon equation in light-cone coordinates. This integral equation can be solved uniquely in terms of the values of the pump and Stokes fields at the entry of the Raman cell. The asymptotic analysis of this solution reveals that the long-distance behavior of the system is dominated by the underlying self-similar solution which satisfies a particular case of the third Painlevé transcendent. This result is consistent with both numerical simulations and experimental observations. We also discuss briefly the effect of frequency mismatch between the pump and the Stokes electric fields.

Key words. Raman cell, Painlevé equations, integrable equations, Riemann-Hilbert problems

MSC numbers. 35, 46

1. Introduction

The Raman cell is a tube, typically a meter long and a couple of centimeters wide, that contains a molecular gas, such as H_2 or D_2. At the entry of the tube, lasers create two incoming electric fields denoted by ε_1 (pump) and ε_2 (Stokes). Let w_1 and w_2 be the frequencies of ε_1 and ε_2, respectively, where $w_1 > w_2$. If $\hbar(w_1 - w_2) = \mathcal{E}$, where $\hbar$ is the Planck constant and $\mathcal{E}$ is the energy between two molecular levels of the gas, then Raman scattering occurs: Either a pump photon is absorbed and a Stokes photon is emitted, or a Stokes photon is emitted and a pump photon is absorbed. Both processes are quantum-mechanicially possible. In either case, the molecule is displaced from level $|1>$ to level $|2>$, and a photon is created. These photons are characterized by a material excitation wave $\mathcal{Q}$ which is proportional to the off-diagonal element ρ_{12} of the associated quantum-mechanical density matrix. This wave is proportional to the spatial correlation of the excited molecules, implying that it is proportional to $\sqrt{N}$, where N is the density of the excited molecules. The field $\mathcal{Q}$ is attenuated on the time scale T_2, the so-called molecular dephasing time. On this time scale, $\mathcal{Q}$ is attenuated because collisions destroy the spatial correlation between excited molecules.

In the model of stimulated Raman scattering considered here, we will neglect a number of effects that appear to be unimportant in the recent experiments that have investigated stimulated Raman scattering in gases [1], [2]: (i) We neglect diffraction. This is valid for the long-focal-length experiments reported in [1] and the references cited therein. (ii) We ignore level saturation, i.e., we assume that only one excitation field is created. This is always valid in molecular gases. (iii) We neglect second and higher-order Stokes generation from spontaneous emission. This assumption is valid only when multipass cells are used [2].

Letting

$$\varepsilon_j = E_j e^{ik_j x - iw_j t}, \qquad j = 1, 2; \qquad \mathcal{Q} = Q e^{ik_3 x - iw_3 t},$$

and making the further assumption that E_1, E_2, Q are slowly varying, it can be shown that the following PDEs are valid:

$$\frac{1}{c}\frac{\partial E_1}{\partial t} + \frac{\partial E_1}{\partial x} = -i\frac{k_1}{k_2}\kappa_2 E_2 Q, \tag{1.1a}$$

$$\frac{1}{c}\frac{\partial E_2}{\partial t} + \frac{\partial E_2}{\partial x} = -i\kappa_2 E_1 \bar{Q}, \tag{1.1b}$$

$$\frac{\partial Q}{\partial t} + \frac{1}{T_2}Q = -i\kappa_1 E_1 \bar{E}_2. \tag{1.1c}$$

In equations (1.1), c is the group velocity of light in the Raman medium, and κ_1, κ_2 are certain constants depending on w_1, w_2 and on the quantum properties of the molecular gas. Also, *throughout this paper, bar denotes complex conjugation*.

Letting

$$\begin{gathered} \chi = \kappa_2 x, \qquad \tau = \kappa_2 t - \frac{\kappa_2}{c}x, \qquad A_1 = \sqrt{\frac{k_2\kappa_1}{k_1\kappa_2}}E_1, \\ A_2 = \sqrt{\tfrac{\kappa_1}{\kappa_2}}E_2, \qquad X = i\sqrt{\tfrac{k_1}{k_2}}Q, \end{gathered} \tag{1.2}$$

and assuming that $T_2\kappa_2 \to \infty$, equations (1.1) become the equations for the transient stimulated Raman scattering,

$$\frac{\partial A_1}{\partial \chi} = -A_2 X, \qquad \frac{\partial A_2}{\partial \chi} = A_1 \bar{X}, \qquad \frac{\partial X}{\partial \tau} = A_1 \bar{A}_2. \tag{1.3}$$

Equations (1.3) imply that $|A_1(\chi,\tau)|^2 + |A_2(\chi,\tau)|^2 = K^2(\tau)$, $K \in \mathbb{R}$. This suggests the introduction of the normalized variables

$$\begin{gathered} A_j' = A_j/K,\ j = 1,2, \qquad X' = X/\tau_\infty, \qquad \chi' = \chi\tau_\infty, \\ \tau_\infty = \int_0^\infty K^2(\tau)d\tau, \qquad \tau' = \int_0^\tau K^2(\xi)\,d\xi. \end{gathered} \tag{1.4}$$

The normalized variables also satisfy equations (1.3) but with $|A_1|^2 + |A_2|^2 = 1$.

The physical problem is specified as follows: $x \in [0, l']$, where l' is the length of the propagation through the Raman cell (which, for a multipass cell, may be many times the cell length), $t \in [0, \infty)$, A_j, $j = 1, 2$, are given at $x = 0$ for all t, and $X = 0$ for all $x \le ct$. Thus, $\tau' \in [0, 1]$, $\chi' \in [0, l]$, $l \doteqdot \kappa_2\tau_\infty l'$, $A_j(0, \tau')$, $j = 1, 2$, are given, and $X(\chi', 0) = 0$. Hence, dropping the primes, we deduce that: *The problem of transient SRS can be formulated as follows: Determine the complex-valued functions $A_1(\chi,\tau)$, $A_2(\chi,\tau)$, $X(\chi,\tau)$ satisfying equations (1.3) with*

$$\tau \in [0,1], \qquad \chi \in [0,l], \qquad |A_1(0,\tau)|^2 + |A_2(0,\tau)|^2 = 1,$$

$$A_j(0,\tau) = A_{j_0}(\tau), \qquad j = 1,2, \qquad X(\chi,0) = 0, \tag{1.5}$$

where $A_{j_0}(\tau)$ are given. *In particular, the interesting physical question is the determination of $A_1(l,\tau)$ and of $A_2(l,\tau)$, where l is large.*

Let the complex-valued function $Y(\chi,\tau)$ and the real-valued function $b(\chi,\tau)$ be defined in terms of $A_1(\chi,\tau)$ and $A_2(\chi,\tau)$ by

$$b = |A_1|^2 - |A_2|^2, \qquad Y = 2iA_1\bar{A}_2. \tag{1.6}$$

If A_1, A_2, X satisfy equations (1.3), then b, Y, X satisfy the equations

$$\frac{\partial b}{\partial \chi} = i(\bar{X}Y - X\bar{Y}), \qquad \frac{\partial Y}{\partial \chi} = 2ibX, \qquad \frac{\partial X}{\partial \tau} = -\frac{i}{2}Y. \tag{1.7}$$

Equations (1.7) are integrable in the sense that they admit a Lax pair formulation. Indeed, it is straightforward to verify that equations (1.7) are the compatibility condition of

$$\frac{\partial \psi}{\partial \chi} = \begin{pmatrix} -ik & X \\ -\bar{X} & ik \end{pmatrix} \psi, \tag{1.8a}$$

and

$$\frac{\partial \psi}{\partial \tau} = \frac{1}{4k}\begin{pmatrix} ib & -Y \\ \bar{Y} & -ib \end{pmatrix} \psi, \tag{1.8b}$$

where $\psi(\chi, \tau, k)$ is a complex-valued 2×2 matrix and k is a complex constant usually referred to as the spectral parameter.

The main aim of this paper is to solve an initial-boundary value problem for equations (1.7), where both χ and τ are in a finite domain. The usual problem of transient stimulated Raman scattering corresponds to $b(0,\tau) = b_0(\tau)$, $Y(0,\tau) = Y_0(\tau)$, $X(\chi, 0) = 0$, where $b_0(\tau)$ and $Y_0(\tau)$ are given functions. Here we shall solve the more general problem of $X(\chi, 0) = X_0(\chi)$, $X_0(\chi)$ given. This corresponds physically to pre-exciting the Raman cell.

Given A_1 and A_2, equations (1.6) determine b and Y uniquely. However, the inverse transformation is not unique. Indeed, if

$$b = \cos\beta, \qquad Y = i\sin\beta e^{-i\theta}, \qquad A_1 = a_1 e^{i\theta_1}, \qquad A_2 = a_2 e^{i\theta_2}, \tag{1.9}$$

then equations (1.6) imply

$$\cos\beta = a_1^2 - a_2^2, \qquad \sin\beta = 2a_1a_2, \qquad \theta = \theta_2 - \theta_1. \tag{1.10}$$

This ambiguity is physically insignificant because one is interested in the phase difference between the pump and the Stokes waves.

We now state the main results of this paper.

Theorem 1.1. *Let $b(\chi,\tau) \in \mathbb{R}$, $Y(\chi,\tau) \in \mathbb{C}$, $X(\chi,\tau) \in \mathbb{C}$ satisfy equations (1.7), with $\chi \in [0, l]$, $l > 0$, and $\tau \in [0, 1]$. Let*

$$b(0,\tau) = b_0(\tau), \qquad Y(0,\tau) = Y_0(\tau), \qquad X(\chi, 0) = X_0(\chi), \tag{1.11a}$$

where $b_0(\tau)$, $Y_0(\tau)$ are differentiable for $\tau \in [0, 1]$, and $X_0(\chi)$ is differentiable for $\chi \in [0, l]$. Assume that

$$b_0(\tau)^2 + |Y_0(\tau)|^2 = 1. \tag{1.11b}$$

The unique solution of this IBV problem is given by

$$\begin{aligned} X(\chi,\tau) &= 2i \lim_{k\to\infty} (k\Psi_1^+(\chi,\tau,k)), \\ b(\chi,\tau) &= -1 - 4i \frac{\partial}{\partial\tau} \lim_{k\to\infty} \left(k\overline{\Psi_2^+(\chi,\tau,\bar{k})}\right), \qquad k \in \mathbb{C}, \quad k_I \neq 0, \end{aligned}$$

where the scalar functions $\Psi_1^+(\chi,\tau,k)$ and $\Psi_2^+(\chi,\tau,k)$, $k \in \mathbb{C}$, can be obtained by solving the following Riemann-Hilbert problem:

$$\begin{pmatrix} \Psi_1^+(\chi,\tau,k) & \frac{\Phi_1^+(\chi,\tau,k)}{\rho_1(k)} \\ \\ \Psi_2^+(\chi,\tau,k) & \frac{\Phi_2^+(\chi,\tau,k)}{\rho_1(k)} \end{pmatrix} = \begin{pmatrix} -\frac{\overline{\Phi_2^+}(\chi,\tau,k)}{\overline{\rho_1}(k)} & \overline{\Psi_2^+}(\chi,\tau,k) \\ \\ \frac{\overline{\Phi_1^+}(\chi,\tau,k)}{\overline{\rho_1}(k)} & -\overline{\Psi_1^+}(\chi,\tau,k) \end{pmatrix}$$

$$\times\begin{pmatrix} 1 & \frac{\rho_2(k)}{\rho_1(k)}e^{2ik\chi+\frac{i\tau}{2k}} \\ -\frac{\overline{\rho_2(k)}}{\overline{\rho_1(k)}}e^{-2ik\chi-\frac{i\tau}{2k}} & \frac{1}{|\rho(k)|^2} \end{pmatrix}, \qquad k\in\mathbb{R}, \tag{1.12a}$$

$$\Phi_1^+(\chi,\tau,k)=1+O\left(\frac{1}{k}\right), \Phi_2^+(\chi,\tau,k)=O\left(\frac{1}{k}\right), \Psi_1^+(\chi,\tau,k)=O\left(\frac{1}{k}\right),$$
$$\Psi_2^+(\chi,\tau,k)=1+O\left(\frac{1}{k}\right), \qquad k\to\infty, \quad k_I\neq 0. \tag{1.12b}$$

This Riemann-Hilbert problem, which is specified through the scalar functions $\rho_1(k)$ and $\rho_2(k)$, $k\in\mathbb{R}$, has a unique solution. The functions $\rho_1(k)$ and $\rho_2(k)$ are constructed as follows: Let $(\mu_1(\tau,k),\mu_2(\tau,k))^T$ be the unique solution of

$$\frac{\partial}{\partial\tau}\begin{pmatrix}\mu_1(\tau,k)\\ \mu_2(\tau,k)\end{pmatrix}=\frac{1}{4k}\begin{pmatrix} ib_0(\tau) & -Y_0(\tau) \\ \bar{Y}_0(\tau) & -ib_0(\tau)\end{pmatrix}\begin{pmatrix}\mu_1(\tau,k)\\ \mu_2(\tau,k)\end{pmatrix}, \tag{1.13a}$$

$$\mu_1(1,k)=1, \qquad \mu_2(1,k)=0. \tag{1.13b}$$

Let $(\nu_1(\chi,k),\nu_2(\chi,k))^T$ be the unique solution of

$$\frac{\partial}{\partial\chi}\begin{pmatrix}\nu_1(\chi,k)\\ \nu_2(\chi,k)\end{pmatrix}=\begin{pmatrix} -ik & X_0(\chi) \\ -\bar{X}_0(\chi) & ik\end{pmatrix}\begin{pmatrix}\nu_1(\chi,k)\\ \nu_2(\chi,k)\end{pmatrix}, \tag{1.14a}$$

$$\nu_1(0,k)=\mu_1(0,k)e^{-\frac{i}{4k}}, \qquad \nu_2(0,k)=\mu_2(0,k)e^{-\frac{i}{4k}}. \tag{1.14b}$$

The functions $\rho_1(k)$ and $\rho_2(k)$ are defined by

$$\rho_1(k)=\nu_1(l,k)e^{ikl}, \qquad \rho_2(k)=\nu_2(l,k)e^{-ikl}. \tag{1.15}$$

If $\rho_1(k)\neq 0$ for $\operatorname{Im} k\geq 0$, the above Riemann-Hilbert problem reduces to solving a system of linear integral equations. In this case, $X(\chi,\tau)$ and $b(\chi,\tau)$ are given by

$$X(\chi,\tau)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\bar{\rho}_2(k)}{\bar{\rho}_1(k)}e^{-2ik\chi-\frac{i\tau}{2k}}M_1(\chi,\tau,k)\,dk,$$

$$b(\chi,\tau)=-1+\frac{2}{\pi}\frac{\partial}{\partial\tau}\int_{-\infty}^{\infty}\frac{\rho_2(k)}{\rho_1(k)}e^{2ik\chi+\frac{i\tau}{2k}}\bar{M}_2(\chi,\tau,k)\,dk, \tag{1.16}$$

where the functions M_1 and M_2 are defined as the unique solution of the following system of linear integral equations:

$$\begin{pmatrix}-M_2(\chi,\tau,k)\\ M_1(\chi,\tau,k)\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{\rho_2(k')}{\rho_1(k')}e^{2ik'\chi+\frac{i\tau}{2k'}}\begin{pmatrix}\bar{M}_1(\chi,\tau,k')\\ \bar{M}_2(\chi,\tau,k')\end{pmatrix}\frac{dk'}{k'-(k-i0)}. \tag{1.17}$$

If $\rho_1(k_j)=0$, $j=1,2,\ldots$, $\operatorname{Im} k_j\geq 0$, then the above Riemann-Hilbert problem reduces to solving a system of linear integral equations similar to (1.17) supplemented by a system of algebraic equations.

The system of algebraic equations needed if $\rho(k)$ has zeros for $\operatorname{Im} k \geq 0$ can be found in [3]. The number of these zeros can be infinite with an accumulation point at $k = 0$ (see the discussion in Section 4). However, these zeros play *no* role in the leading asymptotic behavior of the solution as $\chi \to \infty$, which is dominated by the underlying similarity solution. This is to be contrasted with the usual soliton systems where the number of the zeros is finite and where they dominate the asymptotic behavior of the solution (the finite number of zeros gives rise to a finite number of solitons which determine the leading order behavior of the solution).

The interesting physical question is the computation of $A_1(l, \tau)$ and $A_2(l, \tau)$ (i.e., of $b(l, \tau)$ and of $Y(l, \tau)$) as $l \to \infty$. This is given by the next theorem. For simplicity we concentrate on the more important case of $X_0(\chi) = 0$.

Theorem 1.2. *Consider the IBV problem defined in Theorem 1.1, but with $X_0(\chi) = 0$. The leading order behavior as $\chi \to \infty$ of the solution of this problem is given by*

$$X(\chi, \tau) = \frac{1}{2}\frac{\tau}{\xi}\tilde{X}(\xi), \qquad b(\chi, \tau) = -1 + \frac{1}{4}\frac{\tilde{b}(\xi)}{\xi} + \frac{1}{4}\frac{d}{d\xi}\tilde{b}(\xi), \qquad \xi = \sqrt{\tau\chi}, \tag{1.18}$$

where

$$\tilde{X}(\xi) = -\frac{i}{\pi}\frac{Y_0(0)}{1 - b_0(0)}\int_{-\infty}^{\infty} e^{-i\xi(\lambda + \frac{1}{\lambda})} N_1(\xi, \lambda) d\lambda,$$

$$\tilde{b}(\xi) = \frac{2i}{\pi}\frac{\bar{Y}_0(0)}{1 - b_0(0)}\int_{-\infty}^{\infty} e^{i\xi(\lambda + \frac{1}{\lambda})} \bar{N}_2(\xi, \lambda) d\lambda, \tag{1.19}$$

and the functions N_1, N_2 are the unique solution of the system of linear integral equations

$$\begin{pmatrix} -N_2(\xi, \lambda) \\ N_1(\xi, \lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\pi}\frac{\bar{Y}_0(0)}{1 - b_0(0)}\int_{-\infty}^{\infty} e^{i\xi(\lambda' + \frac{1}{\lambda'})}\begin{pmatrix} \bar{N}_1(\xi, \lambda') \\ \bar{N}_2(\xi, \lambda') \end{pmatrix}\frac{d\lambda'}{\lambda' - (\lambda - i0)}. \tag{1.20}$$

This system is a particular case of a more general system of linear integral equations which characterizes the general solution of Painlevé III equation.

We conclude this introduction with some remarks.

1. The solution defined by equations (1.18)–(1.20) is a similarity solution of equations (1.7). Indeed, if $X(\chi, \tau) = \tau\hat{X}(\xi)$, $Y(\chi, \tau) = \hat{Y}(\xi)$, $b(\chi, \tau) = \hat{b}(\xi)$, $\xi = \sqrt{\chi\tau}$, equations (1.7) reduce to a system of three ODEs for the functions $\hat{X}$, $\hat{Y}$, and $\hat{b}$.
2. A general method for solving IBV problems for nonlinear integrable equations has been introduced recently in [3]. The essence of this method is the introduction of appropriate solutions of both parts of the associated Lax pair that are analytic and bounded for all values of the complex spectral parameter k. It turns out that such solutions are given by $\Phi = \phi \exp[ik\chi\sigma_3 + \frac{i}{4k}(\tau - 1)\sigma_3]$ and $\Psi = \psi \exp[ik\chi\sigma_3 + \frac{i}{4k}\tau\sigma_3]$,

where ϕ and ψ are certain particular solutions of equations (1.8). A significant advantage of this new method is that the spectral data $\rho_1(k)$ and $\rho_2(k)$ are always independent of χ and of τ (see equations (1.15)). As a result of this fact, the Riemann-Hilbert problem characterizing the solution of the given nonlinear equation takes a rather simple form: Its χ and τ dependence is determined by the dispersion relationship of the underlying linearized equation (see equation (1.12a)).

3. The system of linear integral equations (1.20) is the reduction of a certain matrix 2×2 Riemann-Hilbert (RH) problem. The jump for this RH problem occurs on the real axis of the complex λ-plane and it involves the exponential functions $\exp(\pm i\xi(\lambda+\frac{1}{\lambda}))$. This RH problem is a particular case of a more general RH problem, which is associated with the general solution of Painlevé III (PIII) equation [4]. The more general RH problem, in addition to having a "jump" along the real λ-axis, also has a jump along the unit circle of the complex λ-plane. These jumps involve $\exp(\pm i\xi(\lambda+\frac{1}{\lambda}))$ as well as $\lambda^{\pm\theta_0}$ and $\lambda^{\pm\theta_\infty}$, where θ_0 and θ_∞ are constants appearing in PIII equation. In the particular case that $\theta_0 = \theta_\infty = 0$, and the solution is real, the general RH problem associated with PIII reduces to one that can be solved by equation (1.20).
4. The case of frequency mismatch corresponds to the case that the functions $b_0(\tau)$ and/or $Y_0(\tau)$ are singular at $\tau = 0$. For example, consider the case [5] that in the variables defined by equations (1.2), $A_1(0,\tau) = \mathrm{sech}\tau$ and $A_2(0,\tau) = e^{-iw\tau}\mathrm{sech}\tau$. Then in the associated normalized variables, $A_1(0,\tau) = 1/\sqrt{2}$ and $A_2(0,\tau) = \frac{1}{\sqrt{2}}(\frac{\tau}{1-\tau})^{-iw/2}$; thus, $b_0(\tau)$, $Y_0(\tau)$ are singular. It turns out that in this case the spectral data $\rho_1(k)$ and $\rho_2(k)$ are singular at $k = 0$. This gives rise to the more general solution of PIII discussed in (3) above, where $\theta_0 \neq 0$ and $\theta_\infty = 0$. This case is briefly discussed in Section 3.
5. The iteration of the linear integral equation (1.17) shows that this equation involves the integral

$$J(\chi,\tau) = \int_{-\infty}^{\infty} e^{2ik\chi+\frac{i\tau}{2k}} f(k)\, dk. \tag{1.21}$$

The asymptotic evaluation of such integrals is well established (see for example [6]). There exist two important cases. (a) If $\chi \to \infty$ and $\tau/\chi = O(1)$, the stationary phase method implies that $J \sim O(\frac{1}{\sqrt{\chi}})$. This is what happens in the usual soliton systems: The solution of these systems involves a Riemann-Hilbert problem similar to (1.12), which can be reduced to a system of linear integral equations similar to equations (1.17) and to a system of algebraic equations containing the solitonic part of the solution. As $\chi \to \infty$, the contribution from the linear integral equations disperses away and hence the asymptotic behavior of the system is dominated by solitons. (The extension of the stationary phase method from integrals to linear integral equations is given in [7]). (b) If $\chi \to \infty$ and $\tau/\chi = o(1)$, then there exists a moving stationary point and one introduces the *similarity variables* $\xi = \sqrt{\tau\chi}$, $k = \frac{1}{2}\sqrt{\frac{\tau}{\chi}}\lambda$. Then

$$J = \frac{1}{2}\sqrt{\frac{\tau}{\chi}} \int_{-\infty}^{\infty} e^{i\xi(\lambda+\frac{1}{\lambda})} f\left(\frac{1}{2}\sqrt{\frac{\tau}{\chi}}\lambda\right) d\lambda, \tag{1.22}$$

and the leading behavior of the integral depends on the limit of $f(k)$ as $k \to 0$. This case is also relevant in the usual soliton systems but it only characterizes a certain transition zone. It is important to note that in our case $\tau \in [0, 1]$ and $\chi \to \infty$; thus, $\tau/\chi = o(1)$ and the asymptotic behavior of the system is dominated by the underlying similarity solution.

6. The Lax pair of equations (1.7) was found in [8]. Some progress towards the solution of equations (1.7) with $X(\chi, 0) = 0$ was made by Kaup [9] who used the usual inverse scattering method. He bypassed the difficult problem of determining the evaluation of the scattering data by using certain indirect asymptotic arguments. The series of transformations used in [9] makes the relevant analysis rather complicated. As a result, it is difficult to extract the large χ behavior. Nevertheless, one of the authors was able to establish formally that any nonsingular initial data tends towards the similarity solution, using an indirect approach [5]. The approach used here (see the discussion in (1) and (2) above) not only allows one to obtain the key result directly and far more simply, but is also rigorous. Furthermore, it allows one to extend the analysis to the case of singular data (see the discussion in (3) and (4) above). The case that $b(0, \tau)$ and $Y(0, \tau)$ are constant was studied in [10]. Similarity solutions for equations (1.7) were first discussed in [11] (see also [10], [12]). The well-posedness of the stimulated Raman scattering equations is established in [13] using PDE techniques.
7. The inhomogeneously broadened version of equations (1.7) has been solved on the infinite line in [17]. This solution has been used in [18] for the interpretation of the experiments of SRS in gas of [19].

This paper is organized as follows: The linearized version of equations (1.7) is discussed in Section 2. Theorems 1.1 and 1.2 are derived in Section 3. The case of frequency mismatch is briefly discussed in Section 3. The case that $b(0, \tau)$ and $Y(0, \tau)$ are constant is discussed in Section 4. Numerical simulations and experimental observations are discussed in Section 5.

2. The Linearized Equations

In this section we solve the linearized version of the IBV problem considered in Theorem 1.1, namely,

$$Y_\chi = -2iX, \qquad X_\tau = -\frac{i}{2}Y; \qquad \tau \in [0, 1], \qquad \chi \in [0, l], \qquad l > 0, \tag{2.1}$$

$$Y(0, \tau) = Y_0(\tau), \qquad X(\chi, 0) = X_0(\chi), \tag{2.2}$$

where $Y_0(\tau)$ and $X_0(\chi)$ are differentiable functions in $[0, 1]$ and $[0, l]$, respectively. Equations (2.1) imply that Y solves the linearized sine-Gordon equation $Y_{\chi\tau} + Y = 0$.

Equations (2.1) can be obtained from equations (1.7) by assuming that $X = O(\varepsilon)$, $Y = O(\varepsilon)$, and letting $\varepsilon \to 0$. The constraint $|Y|^2 + b^2 = 1$ implies that $b = \pm 1 + O(\varepsilon^2)$; equations (2.1) correspond to $b = -1$.

We shall solve the above IBV problem by using a Lax pair formulation [3]. This has the pedagogical advantage of motivating the formalism used in Section 3. We first *assume* that $Y(\chi, \tau)$ and $X(\chi, \tau)$ exist. After deriving the relevant formulae, we can verify directly that they solve equations (2.1), (2.2) *without* the a priori assumption of existence.

Equations (2.1) admit the Lax pair

$$\frac{\partial \psi}{\partial \chi} + ik\psi = X, \tag{2.3a}$$

$$\frac{\partial \psi}{\partial \tau} + \frac{i}{k}\psi = -\frac{Y}{2k}, \tag{2.3b}$$

where $\psi(\chi, \tau, k)$ is a complex-valued scalar function, and k is a complex parameter. The essence of the method introduced in [3] is to construct solutions of (2.3) that are defined in the entire complex k-plane. Such solutions are

$$\begin{aligned}\psi^{-}(\chi, \tau, k) \ =\ & \frac{e^{-ik\chi}}{2k}\int_{\tau}^{1} e^{-\frac{i}{k}(\tau-\tau')}Y(0, \tau')\,d\tau' \\ & + \int_{0}^{\chi} e^{-ik(\chi-\chi')}X(\chi', \tau)\,d\chi', \quad k_I \le 0,\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\psi^{+}(\chi, \tau, k) \ =\ & -\frac{e^{-ik(\chi-l)}}{2k}\int_{0}^{\tau} e^{-\frac{i}{k}(\tau-\tau')}Y(l, \tau')\,d\tau' \\ & - \int_{\chi}^{l} e^{-ik(\chi-\chi')}X(\chi', \tau)\,d\chi', \qquad k_I \ge 0.\end{aligned} \tag{2.5}$$

The function ψ^{-} defined by equation (2.4) solves equations (2.3) and is analytic and bounded in the lower-half complex k-plane. Indeed, $\tau' - \tau \ge 0$ and $\chi - \chi' \ge 0$, which imply that the exponentials under the first and the second integrals decay as $k \to 0$, $k_I < 0$ and $k \to \infty$, $k_I < 0$, respectively. Also, it is straightforward to verify directly that ψ^{-} solves equations (2.3). Similarly, ψ^{+} is analytic and bounded in the upper-half complex k-plane.

We now indicate how equation (2.4) can be derived: Equation (2.3a) yields

$$\psi(\chi, \tau, k) = e^{-ik\chi}\psi(0, \tau, k) + \int_{0}^{\chi} e^{-ik(\chi-\chi')}X(\chi', \tau)\,d\chi'.$$

The function $\psi(\chi, \tau, k)$ solves equation (2.3b) iff $\psi(0, \tau, k)$ solves equation (2.3b) evaluated at $\tau = 0$; a particular solution of this equation is $\frac{1}{2k}\int_{\tau}^{1} e^{-\frac{i}{k}(\tau-\tau')}Y(0, \tau')\,d\tau'$, and the above equation becomes (2.4). Similarly for (2.5).

Since ψ^{-} and ψ^{+} satisfy both parts of the Lax pair (2.3), it follows that they are related by

$$\psi^{+}(\chi, \tau, k) - \psi^{-}(\chi, \tau, k) = -e^{-ik\chi - \frac{i\tau}{k}}\rho(k), \qquad k \in \mathbb{R}, \tag{2.6}$$

where $\rho(k)$ is some complex-valued scalar function of k. Evaluating equation (2.6) at $\tau = 0$, $\chi = l$, we find

$$\rho(k) = \int_{0}^{l} e^{ik\chi}X(\chi, 0)\,d\chi + \frac{1}{2k}\int_{0}^{1} e^{\frac{i\tau}{k}}Y(0, \tau)\,d\tau. \tag{2.7}$$

Equations (2.4) and (2.5) imply that

$$\psi^{\pm}(\chi,\tau,k) = O\left(\frac{1}{k}\right), \qquad k \to \infty, \qquad k_I \neq 0. \tag{2.8}$$

Equations (2.6) and (2.8) define a Riemann-Hilbert problem [6]. Its unique solution is

$$\psi(\chi,\tau,k) = -\frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{e^{-ik'\chi-\frac{i\tau}{k'}}\rho(k')}{k'-k}\,dk', \qquad k_I \neq 0. \tag{2.9}$$

Equation (2.9) yields ψ in terms of $\rho(k)$. Then equations (2.3), which define X and Y in terms of ψ, yield X and Y in terms of $\rho(k)$. In particular the large k asymptotic of equations (2.3) imply

$$\begin{gathered} X(\chi,\tau) = i\psi^{(1)}(\chi,\tau), \qquad Y(\chi,\tau) = -2\frac{\partial\psi^{(1)}}{\partial\tau}(\chi,\tau), \\ \psi^{(1)}(\chi,\tau) \doteqdot \lim_{\substack{k\to\infty \\ k_I\neq 0}}(k\psi(\chi,\tau,k)). \end{gathered} \tag{2.10}$$

Equations (2.10) yield X and Y in terms of $\rho(k)$, which is uniquely defined in terms of $X_0(\chi)$ and $Y_0(\tau)$. Although these formulae were obtained under the a priori assumption that X and Y exist, it is possible a posteriori to verify directly that the functions X and Y defined in the above way satisfy (2.1) and (2.2). This verification can be found in [3].

Theorem 2.1. [3]. *Let $X_0(\chi)$ and $Y_0(\tau)$ be differentiable functions of χ and of τ for $\chi \in [0,l]$ and $\tau \in [0,1]$. Let $X(\chi,\tau)$ and $Y(\chi,\tau)$ be defined by*

$$X(\chi,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik\chi-\frac{i\tau}{k}}\rho(k)\,dk, \qquad Y(\chi,\tau) = 2i\frac{\partial}{\partial\tau}X(\chi,\tau), \tag{2.11}$$

where $\rho(k)$ is defined by

$$\rho(k) = \int_0^l e^{ik\chi}X_0(\chi)\,d\chi + \frac{1}{2k}\int_0^1 e^{\frac{i\tau}{k}}Y_0(\tau)\,d\tau, \tag{2.12}$$

and improper Riemann integrals are assumed if needed. Then X and Y solve the IBV problem specified by equations (2.1) and (2.2).

3. The Inverse Spectral Method

In order to derive Theorem 1.1, we first assume that $Y(\chi,\tau)$ and $X(\chi,\tau)$ exist. This yields a certain construction of $Y(\chi,\tau)$ and $X(\chi,\tau)$ in terms of the initial and boundary data. We then verify directly that this construction gives rise to $Y(\chi,\tau)$ and $X(\chi,\tau)$, which solve the IBV problem defined in Theorem 1.1 *without* the a priori assumption that $Y(\chi,\tau)$ and $X(\chi,\tau)$ exist. A detailed explanation of all the steps needed can be found in [3].

Proposition 3.1. *Let $b(\chi,\tau)\in\mathbb{R}$, $Y(\chi,\tau)\in\mathbb{C}$, $X(\chi,\tau)\in\mathbb{C}$ be differentiable functions of χ and of τ for $\chi\in[0,l]$, $\tau\in[0,1]$. Assume that Y and X solve equations (1.7). Let the 2×2 matrix complex-valued functions $\Phi(\chi,\tau,k)$ and $\Psi(\chi,\tau,k)$ be defined by*

$$\begin{aligned}
\Phi(\chi,\tau,k) &= e^{-\frac{i}{4k}\left[\int_\tau^1 (b(0,\xi)+1)\,d\xi\right]\sigma_3} \\
&\quad + \frac{e^{-ik\chi\hat{\sigma}_3}}{4k}\int_\tau^1 e^{-\frac{i}{4k}\left(\int_\tau^{\tau'} b(0,\xi)\,d\xi\right)\sigma_3} V(0,\tau')\Phi(0,\tau',k)e^{-\frac{i}{4k}(\tau'-\tau)\sigma_3}d\tau' \\
&\quad + \int_0^\chi e^{-ik(\chi-\chi')\hat{\sigma}_3}U(\chi',\tau)\Phi(\chi',\tau,k)\,d\chi', \qquad (3.1)
\end{aligned}$$

and

$$\begin{aligned}
\Psi(\chi,\tau,k) &= e^{\frac{i}{4k}\left[\int_0^\tau (b(l,\xi)+1)\,d\xi\right]\sigma_3} \\
&\quad - \frac{e^{-ik(\chi-l)\hat{\sigma}_3}}{4k}\int_0^\tau e^{\frac{i}{4k}\left(\int_{\tau'}^{\tau} b(l,\xi)\,d\xi\right)\sigma_3} V(l,\tau')\Psi(l,\tau',k)e^{\frac{i}{4k}(\tau-\tau')\sigma_3}\,d\tau' \\
&\quad - \int_\chi^l e^{-ik(\chi-\chi')\hat{\sigma}_3}U(\chi',\tau)\Psi(\chi',\tau,k)\,d\chi'. \qquad (3.2)
\end{aligned}$$

In equations (3.1) and (3.2),

$$\begin{gathered}
\sigma_3 = \mathrm{diag}(1,-1), \qquad V(\chi,\tau) = \begin{pmatrix} 0 & b(\chi,\tau) \\ -\bar{b}(\chi,\tau) & 0 \end{pmatrix}, \\
U(\chi,\tau) = \begin{pmatrix} 0 & X(\chi,\tau) \\ -\bar{X}(\chi,\tau) & 0 \end{pmatrix},
\end{gathered} \qquad (3.3)$$

and, if A is a 2×2 matrix,

$$\hat{\sigma}_3 A = [\sigma_3, A], \qquad \textit{thus}\ \ e^{\chi\hat{\sigma}_3}A = e^{\chi\sigma_3}Ae^{-\chi\sigma_3}. \qquad (3.4)$$

Then the functions Φ and Ψ have the following properties:

(i) The first column of Φ is analytic and bounded in the upper-half complex-k plane, which will be denoted by $\mathbb{C}^+$, while the second column of Φ is analytic and bounded in the lower-half complex k-plane, which will be denoted by $\mathbb{C}^-$. The function Ψ has complimentary analyticity, i.e., if superscripts $+$ and $-$ denote analyticity in $\mathbb{C}^+$ and $\mathbb{C}^-$, then

$$\Phi = (\Phi^+, \Phi^-), \qquad \Psi = (\Psi^-, \Psi^+). \qquad (3.5)$$

(ii) The functions Φ and Ψ are related by

$$\Phi(\chi,\tau,k) = \Psi(\chi,\tau,k)e^{-i(k\chi+\frac{\tau}{4k})\hat{\sigma}_3}\rho(k), \qquad k\in\mathbb{R}, \qquad (3.6)$$

where the 2×2 *matrix complex-valued function* $\rho(k)$ *is defined by*

$$\begin{aligned}\rho(k) \;=\; & e^{-\frac{i}{4k}\left[\int_0^1 (b(0,\xi)+1)\,d\xi\right]\sigma_3} \\ & + \frac{1}{4k}\int_0^1 e^{-\frac{i}{4k}\left(\int_0^\tau b(0,\xi)\,d\xi\right)\sigma_3} V(0,\tau)\Phi(0,\tau,k)e^{-\frac{i}{4k}\tau\sigma_3}\,d\tau \\ & + \int_0^l e^{ik\chi\hat{\sigma}_3}U(\chi,0)\Phi(\chi,0,k)\,d\chi. \end{aligned} \tag{3.7}$$

(iii) The functions Φ *and* Ψ *satisfy the "symmetry" condition*

$$\begin{aligned} \Phi_2^-(k) &= \overline{\Phi_1^+(\bar{k})}, \qquad & \Phi_1^-(k) &= -\overline{\Phi_2^+(\bar{k})}, \\ \Psi_2^+(k) &= \overline{\Psi_1^-(\bar{k})}, \qquad & \Psi_1^+(k) &= -\overline{\Psi_2^-(\bar{k})}, \end{aligned} \tag{3.8}$$

where for convenience of notation we have suppressed the χ *and* τ *dependence.*

(iv) The asymptotic behavior of Φ *as* $k\to\infty$ *is given by*

$$\begin{aligned}\Phi_1^+(\chi,\tau,k) \;=\; & 1 - \frac{i}{4k}\left[\int_\tau^1 (b(0,\xi)+1)\,d\xi + 2\int_0^\chi |X(\chi',\tau)|^2\,d\chi'\right] \\ & + O\left(\frac{1}{k^2}\right), \qquad k\to\infty, \end{aligned} \tag{3.9}$$

$$\begin{aligned}\Phi_2^+(\chi,\tau,k) \;=\; & \frac{\bar{X}(\chi,\tau)}{2ik} - \frac{1}{4k}\left[\int_\tau^1 \bar{Y}(0,\tau')\,d\tau' - 2i\bar{X}(0,\tau)\right]e^{2ik\chi} \\ & + O\left(\frac{1}{k^2}\right), \qquad k\to\infty. \end{aligned} \tag{3.10}$$

(v) The asymptotic behavior of Φ *as* $k\to 0$ *is given by*

$$\Phi_1^+(\chi,\tau,k) = \alpha_1(\chi,\tau) + \beta_1(\chi,\tau)e^{\frac{i}{2k}(\tau-1)} + O(k), \qquad k\to 0, \tag{3.11}$$

$$\Phi_2^+(\chi,\tau,k) = \alpha_2(\chi,\tau) + \beta_2(\chi,\tau)e^{\frac{i}{2k}(\tau-1)} + O(k), \qquad k\to 0, \tag{3.12}$$

where α_1, α_2, β_1, β_2 *are certain scalar functions of* χ *and* τ.

Proof. Let $\phi(\chi,\tau,k)$ be a 2×2 matrix complex-valued function satisfying the Lax pair (1.8), i.e.,

$$\phi_\chi + ik\sigma_3\phi = U\phi, \tag{3.13a}$$

$$\phi_\tau - \frac{i}{4k}b\sigma_3\phi = -\frac{1}{4k}V\phi. \tag{3.13b}$$

Equation (3.13*a*) implies

$$\phi = e^{-ik\chi\sigma_3}\phi(0,\tau,k) + \int_0^\chi e^{-ik(\chi-\chi')\sigma_3}U(\chi',\tau)\phi(\chi',\tau,k)\,d\chi'.$$

The function $\phi(\chi, \tau, k)$ defined by the above equation will also satisfy equation (3.13b) iff $\phi(0, \tau, k)$ satisfies equation (3.13b) evaluated at $\tau = 0$, i.e., iff $\phi(0, \tau, k)$ satisfies

$$\left(e^{\frac{i}{4k}\breve{b}(0,\tau)\sigma_3}\phi(0, \tau, k)\right)_\tau = -\frac{1}{4k}e^{\frac{i}{4k}\breve{b}(0,\tau)\sigma_3}V(0, \tau)\phi(0, \tau, k),$$

$$\breve{b}(0, \tau) = \int_\tau^1 b(0, \xi)\, d\xi.$$

Thus, if $\phi(\chi, \tau, k)$ is defined by[1]

$$\begin{aligned}\phi \;=\; & e^{-ik\chi\sigma_3 - \frac{i}{4k}\breve{b}(0,\tau)\sigma_3} \\ & + \frac{1}{4k}e^{-ik\chi\sigma_3}\int_\tau^1 e^{-\frac{i}{4k}(\breve{b}(0,\tau)-\breve{b}(0,\tau'))\sigma_3}U(0, \tau')\phi(0, \tau', k)\, d\tau' \\ & + \int_0^\chi e^{-ik(\chi-\chi')\sigma_3}U(\chi', \tau)\phi(\chi', \tau, k)\, d\chi',\end{aligned} \tag{3.14}$$

then ϕ satisfies equations (3.13). Letting

$$\phi = \Phi e^{-ik\chi\sigma_3 + \frac{i}{4k}(1-\tau)\sigma_3}, \tag{3.15}$$

equation (3.14) becomes equation (3.1).

Another solution of equations (3.13) is given by

$$\psi = e^{-ik(\chi-l)\sigma_3}\psi(l, \tau, k) - \int_\chi^l e^{-ik(\chi-\chi')\sigma_3}U(\chi', \tau)\psi(\chi', \tau, k)\, d\chi',$$

where $\psi(l, \tau, k)$ satisfies

$$\left(e^{-\frac{i}{4k}\tilde{b}(l,\tau)\sigma_3}\psi(l, \tau, k)\right)_\tau = -\frac{1}{4k}e^{-\frac{i}{4k}\tilde{b}(l,\tau)\sigma_3}V(l, \tau)\psi(l, \tau, k),$$

$$\tilde{b}(l, \tau) = \int_0^\tau b(l, \xi)\, d\xi.$$

Thus, if $\psi(\chi, \tau, k)$ is defined by

$$\begin{aligned}\psi \;=\; & e^{-ik\chi\sigma_3 + \frac{i}{4k}\tilde{b}(l,\tau)\sigma_3} \\ & - \frac{1}{4k}e^{-ik(\chi-l)\sigma_3}\int_0^\tau e^{\frac{i}{4k}(\tilde{b}(l,\tau)-\tilde{b}(l,\tau'))\sigma_3}U(l, \tau')\psi(l, \tau', k)\, d\tau' \\ & - \int_l^\chi e^{-ik(\chi-\chi')\sigma_3}U(\chi', \tau)\psi(\chi', \tau, k)\, d\chi',\end{aligned} \tag{3.16}$$

then ψ satisfies equations (3.13). Letting

$$\psi = \Psi e^{-ik\chi\sigma_3 - \frac{i\tau}{4k}\sigma_3}, \tag{3.17}$$

equation (3.16) becomes equation (3.2).

---

[1] We have chosen the constant of integration to be the identity matrix.

(i) The Relationship between ϕ and ψ

Since ϕ and ψ satisfy both parts of the Lax pair, it follows that $\phi = \psi\rho(k)$. Indeed, since ϕ and ψ satisfy equation (3.13a), then $\phi = \psi f(k, t)$, where f is a 2×2 matrix. Similarly, since ϕ and ψ satisfy equation (3.13b), then $\phi = \psi g(k, \chi)$; thus, $\phi = \psi\rho(k)$. Using (3.15) and (3.17), this equation becomes (3.6), where $\rho(k)$ is some 2×2 matrix. Evaluating equation (3.6) at $\chi = l$, $\tau = 0$, we find that $\rho(k) = e^{ikl\hat{\sigma}_3}\Phi(l, 0, k)$, which yields equation (3.7).

(ii) The Symmetry Properties

Equations (3.8) are a direct consequence of $\bar{V}^T = -V$, $\bar{U}^T = -U$, where T denotes transpose.

(iii) The Analyticity Properties

The functions Φ and Ψ are entire functions of k for all complex k except possibly 0 and ∞. In what follows we concentrate on Φ; analogous results for Ψ can be obtained in a similar manner.

In order to determine the behavior of Φ as $k \to \infty$, we note that if A is a 2×2 matrix, then

$$e^{-ik(\chi-\chi')\hat{\sigma}_3}A = \begin{pmatrix} A_{11} & A_{12}e^{-2ik(\chi-\chi')} \\ A_{21}e^{2ik(\chi-\chi')} & A_{22} \end{pmatrix}.$$

Thus the exponential terms of the first column in the integral $\int_0^\chi$ decay as $k \to \infty$, $k_I > 0$ (the exponential terms of the second column decay as $k \to \infty$, $k_I < 0$). Since $\chi \geq 0$, similar considerations apply to the term $e^{-ik\chi\hat{\sigma}_3}$.

In order to determine the behavior of Φ as $k \to 0$, we note that

$$e^{-\frac{i}{4k}\left(\int_\tau^{\tau'} b(0,\xi)\,d\xi\right)\sigma_3} A e^{-\frac{i}{4k}(\tau'-\tau)\sigma_3}$$
$$= \begin{pmatrix} e^{-\frac{i}{4k}\int_\tau^{\tau'}(b(0,\xi)+1)\,d\xi}A_{11} & e^{\frac{i}{4k}\int_\tau^{\tau'}(1-b(0,\xi))\,d\xi}A_{12} \\ e^{-\frac{i}{4k}\int_\tau^{\tau'}(1-b(0,\xi))\,d\xi}A_{21} & e^{\frac{i}{4k}\int_\tau^{\tau'}(b(0,\xi)+1)\,d\xi}A_{22} \end{pmatrix}.$$

Since $|b(0, \xi)| < 1$, it follows that $\int_\tau^{\tau'}(1 \pm b(0, \xi))\,d\xi > 0$ for $\tau' > \tau$, and hence the exponential terms of the first column in the integral $\int_\tau^1$ decay as $k \to 0$, $k_I > 0$. Since $\tau \leq 1$, similar considerations apply to the term $e^{-\frac{i}{4k}[\int_\tau^1(b(0,\xi)+1)\,d\xi]}$.

The leading behavior of Φ for $k \to \infty$ and $k \to 0$ is determined below.

(iv) The Large k Behavior

The easiest way to determine the behavior of Φ when $k \to \infty$ and $k_I \neq 0$ is to use equations (3.13) and equation (3.15). These equations imply that

$$\Phi_\chi + ik[\sigma_3, \Phi] - U\Phi = 0, \tag{3.18a}$$

$$\Phi_\tau - \frac{i}{4k}\Phi\sigma_3 - \frac{i}{4k}b\sigma_3\Phi + \frac{1}{4k}V\Phi = 0. \tag{3.18b}$$

Substituting $\Phi = I + \Phi^{(1)}(\chi, \tau)/k + O(k^{-2})$ into these equations, we find

$$\Phi_{12}^{(1)} = \frac{X}{2i}, \qquad \Phi_{21}^{(1)} = \frac{\bar{X}}{2i}, \qquad \Phi_{11\chi}^{(1)} = \frac{|X|^2}{2i},$$

$$\Phi_{11\tau}^{(1)} = \frac{i}{4}(b+1), \qquad \Phi_{22}^{(1)} = -\Phi_{11}^{(1)}.$$

Using equations (1.7a) and (1.7c), it follows that the equations for Φ_{11} are compatible. Also the above equations are consistent with equations (3.9) and (3.10) (for $k_I \neq 0$, the term involving $e^{2ik\chi}$ is exponentially small).

To determine the precise behavior of Φ for all large k, we use equation (3.1). Substituting

$$\Phi_1^+ = 1 + \frac{\tilde{\alpha}_1(\chi, \tau)}{k} + O\left(\frac{1}{k^2}\right), \qquad \Phi_2^+ = \frac{\tilde{\alpha}_2(\chi, \tau)}{k} + \frac{\tilde{\beta}_2(\chi, \tau)}{k}e^{2ik\chi} + O\left(\frac{1}{k^2}\right),$$

into (3.1) and using integration by parts, equations (3.9), (3.10) follow.

(v) The Small k Behavior

The easiest way to determine the behavior of Φ when $k \to 0$ and $k_I \neq 0$ is to use equations (3.18). Substituting

$$\Phi(\chi, \tau, k) = \Phi^{(0)}(\chi, \tau) + k\Phi^{(1)}(\chi, \tau) + O(k^2)$$

in equations (3.18), it follows that $\Phi_\chi^{(0)} = U\Phi^{(0)}$ and

$$\Phi^{(0)}(\chi, \tau)\sigma_3\left(\Phi^{(0)}(\chi, \tau)\right)^{-1} = \begin{pmatrix} -b(\chi, \tau) & -iY(\chi, \tau) \\ i\bar{Y}(\chi, \tau) & b(\chi, \tau) \end{pmatrix}.$$

This equation can be solved for $\Phi^{(0)}$ since the determinant of the lhs is -1, while the determinant of the rhs is $-b^2 - |Y|^2 = -1$. Actually, using the above equation together with $\det \Phi^{(0)} = 1$, it follows that

$$\Phi_{11}^{(0)} = \frac{1-b}{2\Phi_{22}^{(0)}}, \qquad \Phi_{12}^{(0)} = \frac{iY}{1-b}\Phi_{22}^{(0)}, \qquad \Phi_{21}^{(0)} = \frac{i\bar{Y}}{2\Phi_{22}^{(0)}}. \tag{3.19}$$

To determine the precise behavior of Φ for all small k, we use equation (3.1); see Appendix A. □

Derivation of Theorem 1.1

(i) Definition of ρ(k)

Proposition 3.1 suggests that the spectral data $\rho(k)$ should be defined by equation (3.7), which involves the known functions $b(0,\tau)$, $V(0,\tau)$, and $U(\chi,0)$, as well as $\Phi(0,\tau,k)$ and $\Phi(\chi,0,k)$.

We define the function $\Phi(0,\tau,k)$ by

$$\begin{aligned}\Phi(0,\tau,k) \;&=\; e^{-\frac{i}{4k}\left[\int_\tau^1 (b(0,\xi)+1)\,d\xi\right]\sigma_3} \\ &\quad + \frac{1}{4k}\int_\tau^1 e^{-\frac{i}{4k}\left(\int_\tau^{\tau'} b(0,\xi)\,d\xi\right)\sigma_3} V(0,\tau')\Phi(0,\tau',k)e^{-\frac{i}{4k}(\tau'-\tau)\sigma_3}\,d\tau'.\end{aligned} \tag{3.20}$$

This function is the unique solution of

$$\Phi_\tau - \frac{i}{4k}\Phi\sigma_3 - \frac{i}{4k}b(0,\tau)\sigma_3\Phi = -\frac{1}{4k}V(0,\tau)\Phi, \tag{3.21a}$$

$$\Phi(0,1,k) = I. \tag{3.21b}$$

Letting

$$\Phi(0,\tau,k) = \mu(\tau,k)e^{\frac{i}{4k}(\tau-1)\sigma_3}, \tag{3.22}$$

it follows that $\mu(\tau,k)$ is the unique solution of

$$\mu_\tau - \frac{i}{4k}b(0,\tau)\sigma_3\mu = -\frac{1}{4k}V(0,\tau)\mu, \tag{3.23a}$$

$$\mu(1,k) = I. \tag{3.23b}$$

We define the function $\Phi(\chi,0,k)$ by the equation

$$\begin{aligned}\Phi(\chi,0,k) \;&=\; e^{-\frac{i}{4k}\left[\int_0^1 (b(0,\xi)+1)\,d\xi\right]\sigma_3} \\ &\quad + \frac{e^{-ik\chi\hat{\sigma}_3}}{4k}\int_0^1 e^{-\frac{i}{4k}\left(\int_0^{\tau} b(0,\xi)\,d\xi\right)\sigma_3} V(0,\tau)\Phi(0,\tau,k)e^{-\frac{i}{4k}\tau\sigma_3}\,d\tau \\ &\quad + \int_0^\chi e^{-ik(\chi-\chi')\hat{\sigma}_3}U(\chi',0)\Phi(\chi',0,k)\,d\chi'.\end{aligned} \tag{3.24}$$

This function is the unique solution of

$$\Phi_\chi + ik[\sigma_3,\Phi] = U(\chi,0)\Phi, \tag{3.25a}$$

$$\Phi(0,0,k) = \mu(0,k)e^{-\frac{i}{4k}\sigma_3}. \tag{3.25b}$$

Letting

$$\Phi(\chi,0,k) = \nu(\chi,k)e^{ik\chi\sigma_3}, \tag{3.26}$$

it follows that $\nu(\chi, k)$ is the unique solution of

$$\nu_\chi + ik\sigma_3\nu = U(\chi, 0)\nu, \tag{3.27a}$$

$$\nu(0, k) = \mu(0, k)e^{-\frac{i}{4k}\sigma_3}. \tag{3.27b}$$

Having obtained $\nu(\chi, k)$, $\rho(k)$ follows from

$$\rho(k) = e^{ikl\hat{\sigma}_3}\Phi(l, 0, k) = e^{ikl\sigma_3}\nu(l, k). \tag{3.28}$$

Using the symmetries of V and U, it follows that

$$\mu(\tau, k) = \begin{pmatrix} \mu_1(\tau, k) & -\overline{\mu_2(\tau, \bar{k})} \\ \mu_2(\tau, k) & \overline{\mu_1(\tau, \bar{k})} \end{pmatrix}, \qquad \nu(\chi, k) = \begin{pmatrix} \nu_1(\chi, k) & -\overline{\nu_2(\chi, \bar{k})} \\ \nu_2(\chi, k) & \overline{\nu_2(\chi, \bar{k})} \end{pmatrix},$$

$$\rho(k) = \begin{pmatrix} \rho_1(k) & -\overline{\rho_2(\bar{k})} \\ \rho_2(k) & \overline{\rho_2(\bar{k})} \end{pmatrix}.$$

Thus,

$$\rho_1(k) = e^{ikl}\nu_1(l, k), \qquad \rho_2(k) = e^{-ikl}\nu_2(l, k),$$

which are equations (1.15).

Using equations (3.20), (3.24) and integration by parts, it is straightforward to obtain the large k and small k behavior of $\rho(k)$. The derivation is similar to the one used to derive equations (3.9)–(3.12); see Appendix B.

$$\begin{aligned}\rho_2(k) \;=\; & 1 - \frac{i}{4k}\left[\int_0^1 (b(0, \xi) + 1)\, d\xi + 2\int_0^l |X(\chi, 0)|^2\, d\xi\right] \\ & + O\left(\frac{1}{k^2}\right), \qquad k \to \infty,\end{aligned} \tag{3.29a}$$

$$\rho_2(k) = \frac{\bar{X}(l, 0)}{2ik}e^{-2ikl} - \frac{1}{4k}\left[\int_0^1 \bar{Y}(0, \tau)\, d\tau - 2i\bar{X}(0, 0)\right] + O\left(\frac{1}{k^2}\right), \quad k \to \infty. \tag{3.29b}$$

$$\rho_1(k) = \alpha_1 + \beta_1 e^{-\frac{i}{2k}} + O(k), \qquad k \to 0 \tag{3.30a}$$

$$\rho_2(k) = \alpha_2 + \beta_2 e^{-\frac{i}{2k}} + O(k), \qquad k \to 0, \tag{3.30b}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are certain constants.

(ii) The RH Problem

Equations (3.6) suggest that the functions Φ and Ψ should be defined as the unique solution of the following Riemann-Hilbert (RH) problem:

$$\begin{aligned}(\Phi^+(\chi,\tau,k),\ &\Phi^-(\chi,\tau,k))\\ &=(\Psi^-(\chi,\tau,k),\Psi^+(\chi,\tau,k))\\ &\quad\times\begin{pmatrix}\rho_1(k) & -\bar{\rho}_2(k)e^{-2ik\chi-\frac{i\tau}{2k}}\\ \rho_2(k)e^{2ik\chi+\frac{i\tau}{2k}} & \bar{\rho}_1(k)\end{pmatrix},\qquad k\in\mathbb{R}\end{aligned}\tag{3.31a}$$

$$\Phi=I+O\left(\frac{1}{k}\right),\qquad \Psi=I+O\left(\frac{1}{k}\right),\qquad k\to\infty\ \text{with}\ k_I\neq 0.\tag{3.31b}$$

This RH problem is identical to the one associated with the sine-Gordon equation in light-cone coordinates (see equations (4.16) of [3] with x and t replaced by χ and τ, respectively). Details of the analysis of this RH problem can be found in [3]. Here we only summarize the main points.

(a) Equation (3.31a) can be rewritten as

$$\left(\Psi^+,\frac{\Phi^+}{\rho_1}\right)=\left(\frac{\Phi^-}{\bar{\rho}_1},\Psi^-\right)\begin{pmatrix}1 & \frac{\rho_2}{\rho_1}E\\ -\frac{\bar{\rho}_2}{\bar{\rho}_1}\bar{E} & \frac{1}{|\rho_1|^2}\end{pmatrix},\qquad E=e^{2ik\chi+\frac{i\tau}{2k}},\qquad k\in\mathbb{R}.\tag{3.32}$$

Assuming that $\rho_1(k)\neq 0$ for $k\in\mathbb{C}^+$, the first vector of the above equation implies (taking the complex conjugate and the plus projection)

$$\begin{aligned}&\overline{\Psi^+}(\chi,\tau,k)\\ &\quad=\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{\rho_2(k')}{\rho_1(k')}E(\chi,\tau,k')\overline{\Psi^-}(\chi,\tau,k')\frac{dk'}{k'-(k-i0)},\quad k\in\mathbb{R}.\end{aligned}\tag{3.33}$$

Using the symmetry conditions $\Psi_2^-=-\overline{\Psi_1^+}$, $\Psi_1^-=\overline{\Psi_2^+}$, equation (3.33) becomes equation (1.17) (we have called $M_1=\Psi_1^-$, $M_2=\Psi_2^-$).

The equations

$$\Phi_{12}^{(1)}=\frac{X}{2i},\qquad \Phi_{11_\tau}^{(1)}=\frac{i}{4}(b+1)$$

imply

$$X=2i\lim_{k\to\infty}(k\Psi_1^+),\qquad b=-1-4i\frac{\partial}{\partial\tau}\lim_{k\to\infty}(k\bar{\Psi}_2^+).$$

(b) Equation (1.17) can be solved uniquely *without* having to assume that ρ_2/ρ_1 is small. This is a consequence of the fact that the jump matrix G appearing in equation (3.32) satisfies $\bar{G}^T=-G$. This can be used to show that the homogeneous version of the RH problem with the jump (3.32) has only the trivial solution [14].

(c) The case that $\rho_1(k) = 0$ for $k = k_j,\ j = 1, 2, \ldots, Im\, k_j > 0$, can be reduced to solving a regular RH problem (i.e., a RH problem with $\rho_1(k) \neq 0$) and a system of algebraic equations. This system of algebraic equations can be found in [3].

(iii) The Inverse Problem Solves the Direct

We must show that the solution of the above RH problem satisfies the Lax pair (3.18). Furthermore, we must show that the functions $b(\chi,\tau)$, $Y(\chi,\tau)$, $X(\chi,\tau)$ defined by equations (1.16) solve the IBV problem defined in Theorem 1.1. This involves the use of the so-called dressing method (see [15] for the rigorous implementation of this method): Let $L_1\Phi$ and $L_2\Phi$ denote the lhs's of equations (3.18a) and (3.18b), respectively. The main idea of the dressing method is the following: (a) Define b, Y, X in terms of Φ in such a way that the $O(k)$, $O(1)$, $O(\frac{1}{k})$ terms of $L_1\Phi$ and of $L_2\Phi$ are zero. (b) Show that both $L_1\Phi$ and $L_2\Phi$ satisfy the jump condition (3.31a). Since the homogeneous version of the RH problem (3.31) admits only the zero solution, this implies that $L_1\Phi = L_2\Phi = 0$.

Details of (a) are given in Appendix C. It is shown there that $X = 2i(\Psi_1^+)^{(1)}$, $b = -1-4i(\bar{\Psi}_2^+)_\tau^{(1)}$, where $(\Psi_1^+)^1$, $(\bar{\Psi}_2^+)^1$ are the $O(\frac{1}{k})$ terms of Ψ_1^+ and $\bar{\Psi}_2^+$. These equations, together with equation (3.33), yield equations (1.16).

Derivation of Theorem 1.2

Let

$$k = \frac{1}{2}\sqrt{\frac{\tau}{\chi}}\lambda, \qquad \xi = \sqrt{\tau\chi}.$$

Then the kernel of equation (3.33) becomes

$$\frac{\rho_2\left(\frac{1}{2}\sqrt{\frac{\tau}{\chi}}\lambda'\right)}{\rho_1\left(\frac{1}{2}\sqrt{\frac{\tau}{\chi}}\lambda'\right)} e^{i\xi\left(\lambda'+\frac{1}{\lambda'}\right)} \frac{d\lambda'}{\lambda'-(\lambda-i0)}.$$

Thus the leading order behavior of equation (3.33) as $\chi \to \infty$ depends on the limit of $\rho_2(k)/\rho_1(k)$ as $k \to 0$. Equations (3.30) yield

$$\frac{\rho_2(k)}{\rho_1(k)} \sim \frac{\alpha_2 + \beta_2 e^{-\frac{i}{2k}}}{\alpha_1 + \beta_1 e^{-\frac{i}{2k}}}, \qquad k \to 0,$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2$ are certain constants. It is interesting that the terms involving $\exp(-i/2k)$ give no contribution. Indeed,

$$\frac{\rho_2(k)}{\rho_1(k)} e^{2ik\chi+\frac{i\tau}{2k}} \sim \frac{\alpha_2}{\alpha_1}\left[e^{2ik\chi+\frac{i\tau}{2k}} + \frac{\left(\frac{\beta_2}{\alpha_2}-\frac{\beta_1}{\alpha_1}\right)e^{2ik\chi-\frac{i}{2k}(1-\tau)}}{1+\frac{\beta_1}{\alpha_1}e^{-\frac{i}{2k}}}\right], \qquad k \to 0.$$

If $\alpha_1 + \beta_1 e^{-\frac{i}{2k}} \neq 0$ for $\mathrm{Im}\, k \geq 0$, then because of analyticity in $\mathbb{C}^+$ (since $\tau \leq 1$, the exponential terms decay in $\mathbb{C}^+$), these terms give zero contribution to equation (3.33)

($\overline{\Psi^-}$ and $[k' - (k - i0)]^{-1}$ are also analytic in $\mathbb{C}^+$). If $\alpha_1 + \beta_1 e^{-\frac{i}{2k}} = 0$, the extra terms due to the poles give a contribution that is exponentially small as $\chi \to \infty$.

The above analysis implies that the leading behavior of equation (3.33) as $\chi \to \infty$ is characterized by

$$\overline{\Psi^+}(\xi, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2i\pi} \frac{\alpha_2}{\alpha_1} \int_{-\infty}^{\infty} e^{i\xi\left(\lambda' + \frac{1}{\lambda'}\right)} \overline{\Psi^-}(\xi, \lambda') \frac{d\lambda'}{\lambda' - (\lambda - i0)}. \tag{3.34}$$

We emphasize that even if $\rho(k)$ has zeros for $Im\, k > 0$, these zeros do not contribute to the leading behavior of the solution of the Riemann-Hilbert problem (1.12). Indeed, if there exist zeros, equation (3.33) has to be supplemented by certain additional terms. However, these terms vanish exponentially as $\chi \to \infty$. Consider for simplicity the case of one zero, $\rho(k_1) = 0$; the extension to any number of zeros is straightforward. If $\rho(k_1) = 0$, $\mathrm{Im}\, k_1 > 0$, the rhs of equation (3.33) also contains the term

$$-\frac{\rho_2(k_1) e^{2ik_1\chi + \frac{i\tau}{2k_1}} \overline{\Psi^-(k_1)}}{\dot\rho(k_1)(k - k_1)}, \qquad \dot\rho(k_1) = \left.\frac{d\rho}{dk}\right|_{k=k_1},$$

where $\Psi^-(k_1)$ is given in terms of a certain linear integral equation whose kernel involves $(k' - k_1)^{-1}$. Since the term $e^{2ik_1\chi}$, $Im\, k_1 > 0$ is exponentially small, it follows that the zeros give an exponentially small contribution as $\chi \to \infty$.

The above analysis is valid even if $X(\chi, 0) \neq 0$. In the particular case that $X(\chi, 0) = 0$, equations (B.10) and (B.11) yield

$$\alpha_2 = \frac{i\bar{Y}(0,0)}{1 - b(0,0)} \alpha_1. \tag{3.35}$$

Equations (3.34), (3.35), together with the symmetry conditions $\Psi_2^- = -\overline{\Psi_1^+}$, $\Psi_1^- = \overline{\Psi_2^+}$, yield equation (1.20) (we have called $N_1 = \Psi_1^-$, $N_2 = \Psi_2^-$). Equations (1.16) imply

$$X = \frac{1}{2}\sqrt{\frac{\tau}{\chi}}\tilde{X}(\xi), \qquad b = -1 + \frac{\partial}{\partial\tau}\left(\frac{1}{2}\sqrt{\frac{\tau}{\chi}}\tilde{b}(\xi)\right), \tag{3.36}$$

where $\tilde{X}(\xi)$ and $\tilde{b}(\xi)$ are defined by (1.19) (since $\rho_2/\rho_1 \sim \alpha_2/\alpha_1 = i\bar{Y}/(1-b)$). Using

$$\sqrt{\frac{\tau}{\chi}} = \frac{\tau}{\xi}, \qquad \frac{\partial}{\partial\tau} = \frac{1}{2}\sqrt{\frac{\chi}{\tau}}\frac{\partial}{\partial\xi},$$

equations (3.36) become equations (1.18).

The Case of Frequency Mismatch

When there exists a frequency mismatch between the physical quantities A_1 and A_2, then the transformation (1.4) induces a singularity on the transformed quantities A_1' and A_2'. For example, if

$$A_1 = \mathrm{sech}(\tau + \tau_0), \qquad A_2 = e^{-i\omega\tau}\mathrm{sech}(\tau + \tau_0),$$

then, with an appropriate choice of τ_0,

$$A_1' = \frac{1}{2}, \qquad A_2' = \frac{1}{2}\left(\frac{\tau'}{1-\tau'}\right)^{-i\omega\tau'}.$$

The mismatch between A_1 and A_2 occurs in physical systems because the frequency difference between the pump and Stokes waves does not precisely match the frequency difference between the Raman levels (to minimize this effect, the experimentalists physically connect the two cells but this effect can never be completely eliminated).

The rigorous analysis for the case of singular data is rather technical, so here we only present a brief summary of the main ideas.

Consider first the linear equations (2.1) with the initial and boundary conditions

$$X_0(\chi) = 0, \qquad Y_0(\tau) = c\tau^{\gamma} + o(\tau^{\gamma}) \text{ as } \tau \to 0^+, \qquad \gamma \in \mathbb{R}, \qquad \gamma > -1, \tag{3.37}$$

where c is a constant. Using the analogue of Watson's lemma for Fourier-type integrals, it can be shown (see for example [6]) that

$$\rho(k) = \frac{1}{2k}\int_0^1 e^{\frac{i\tau}{k}} Y_0(\tau)\,d\tau = ck^{\gamma}\frac{\Gamma(\gamma+1)}{2}e^{\frac{i\pi}{2}(\gamma+1)}[1+o(k^{\gamma+1})], \qquad k \to 0, \tag{3.38}$$

where Γ denotes the gamma function. Furthermore, equations (2.4) and (2.5) indicate that ψ^- and ψ^+ are also singular at $k = 0$.

Similarly, regarding the nonlinear equation (1.7), the eigenfunctions Φ and Ψ, as well as the matrix $\rho(k)$ containing the spectral data, are singular at $k = 0$. Since Φ and Ψ are well defined away from $k = 0$, it is convenient to define Φ and Ψ for k such that $|k| \geq 1$, and to introduce two new eigenfunctions Φ_0 and Ψ_0 that are defined for $|k| \leq 1$. This gives rise to a new RH problem whose "jumps" in the complex k-plane occur along the real axis and along the unit circle.

The large χ behavior of the above RH problem can be obtained by using the following substitutions:

$$\begin{gathered} k = \frac{1}{2}\sqrt{\frac{\tau}{\chi}}\lambda, \qquad \xi = \sqrt{\tau\chi}, \qquad \Phi(\chi,\tau,k) = \hat{\Phi}(\xi,\lambda), \\ \Phi_0(\chi,\tau,k) = \hat{\Phi}_0(\xi,\lambda)\left(\frac{1}{2}\sqrt{\frac{\tau}{\chi}}\right)^{-\gamma\sigma_3}. \end{gathered} \tag{3.39}$$

Using equations (3.39), together with the fact that

$$\rho(k) \sim k^{\gamma\sigma_3}, \qquad k \to 0,$$

the above RH problem reduces to a RH problem with the following jump condition:

$$\Phi^+(\xi,\lambda) = \Phi^-(\xi,\lambda)e^{i\xi(\lambda+\frac{1}{\lambda})\sigma_3}V(\lambda)e^{-i\xi(\lambda+\frac{1}{\lambda})\sigma_3}, \qquad \lambda \in \mathbb{R}\cup\{|\lambda|=1\}.$$

The matrix $V(\lambda)$ is given by (see Figure 1)

$$V_{\infty A} = (G_1^{\infty})^{-1}, \qquad V_{\overset{\frown}{AB}} = \lambda^{\gamma\sigma_3}E, \qquad V_{\infty B} = (G_2^{\infty})^{-1},$$

$$V_{\underset{\frown}{AB}} = \lambda^{\gamma\sigma_3}(G_1^0)^{-1}EG_1^{\infty},$$

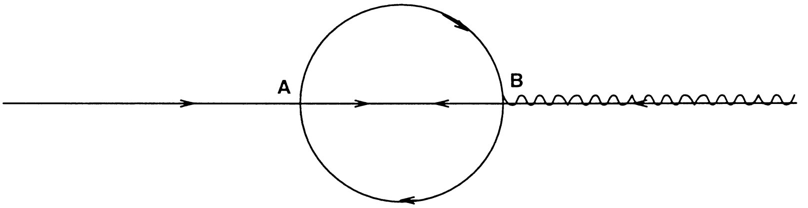

Fig. 1. The Riemann-Hilbert problem for the case of frequency mismatch.

$$V_{AO} = \lambda^{\gamma\sigma_3}(G_1^0)^{-1}\lambda^{-\gamma\sigma_3}, \qquad V_{BO} = (\lambda^{\gamma\sigma_3})_+(G_2^0)^{-1}(\lambda^{-\gamma\sigma_3})_+,$$

where the subscript $+$ in V_{BO} indicates that $\lambda^{\gamma\sigma_3}$ is evaluated by considering the limit from the $+$ region, i.e., $(\lambda)_+ = |\lambda|e^{2i\pi}$. The constant matrices G_1^∞, G_2^∞, G_1^0, G_2^0 are defined by

$$G_1^0 = \begin{pmatrix} 1 & a_0 \\ 0 & 1 \end{pmatrix}, \qquad G_2^0 = \begin{pmatrix} 1 & 0 \\ b_0 & 1 \end{pmatrix}, \qquad G_1^\infty = \begin{pmatrix} 1 & a_\infty \\ 0 & 1 \end{pmatrix}, \qquad G_2^\infty = \begin{pmatrix} 1 & 0 \\ b_\infty & 1 \end{pmatrix},$$

and E satisfies $G_1^\infty G_2^\infty e^{-2i\pi\gamma\sigma_3} = E^{-1}G_1^0 G_2^0 E^{2i\pi\gamma\sigma_3}E$. The constant scalars a_0, b_0, a_∞, b_∞ can be evaluated in terms of $Y_0(\tau)$ and $b_0(\tau)$.

Having obtained $\Phi(\xi, \lambda)$, the leading behavior of X and b follows from

$$X(\chi, \tau) = \frac{1}{2}\frac{\tau}{\xi}\tilde{X}(\xi), \qquad b(\chi, \tau) = -1 + \frac{1}{4}\frac{\tilde{b}(\xi)}{\xi} + \frac{1}{4}\frac{d}{d\xi}\tilde{b}(\xi),$$

where $\tilde{X}(\xi)$ and $\tilde{b}(\xi)$ can be obtained from the large λ behavior of $\Phi(\xi, \lambda)$.

The above RH problem gives rise to a solution of Painlevé III

$$\frac{d^2u}{dt^2} = \frac{1}{u}\left(\frac{du}{dt}\right)^2 - \frac{1}{t}\frac{du}{dt} + \frac{1}{t}(\alpha_1 u^2 + \alpha_2) + \alpha_3 u^3 + \frac{\alpha_4}{u},$$

with the following particular values of the constants α_j, $j = 1, \ldots, 4$:

$$\alpha_1 = 2\gamma, \qquad \alpha_2 = \alpha_3 = -\alpha_4 = 4.$$

4. A Particular Example

In this section we consider the particular example,

$$X(\chi, 0) = 0, \qquad b(0, \tau) = \beta \in \mathbb{R}, \qquad Y(0, \tau) = \gamma \in \mathbb{C}, \qquad \beta^2 + |\gamma|^2 = 1, \tag{4.1}$$

where β and γ are constants. This example corresponds to data which are initially self-similar. Thus, it plays a role somewhat analogous to the hyperbolic secant solution in the usual soliton systems.

Equations (1.13) yield

$$\mu_1(\tau,k) = \cos\left(\frac{\tau-1}{4k}\right) + i\beta\sin\left(\frac{\tau-1}{4k}\right), \tag{4.2a}$$

$$\mu_2(\tau,k) = \bar{\gamma}\sin\left(\frac{\tau-1}{4k}\right). \tag{4.2b}$$

Equations (1.14) yield

$$\nu_1(\chi,k) = \mu_1(0,k)e^{-\frac{i}{4k}-ik\chi}, \qquad \nu_2(\chi,k) = \mu_2(0,k)e^{-\frac{i}{4k}+ik\chi}.$$

Equations (1.15) yield

$$\rho_1(k) = \mu_1(0,k)e^{-\frac{i}{4k}}, \qquad \rho_2(k) = \mu_2(0,k)e^{-\frac{i}{4k}},$$

or, using (4.2),

$$\rho_1(k) = \frac{1-\beta}{2} + \frac{1+\beta}{2}e^{-\frac{i}{2k}}, \tag{4.3}$$

$$\rho_2(k) = -\frac{\bar{\gamma}}{2i} + \frac{\bar{\gamma}}{2i}e^{-\frac{i}{2k}}. \tag{4.4}$$

The terms involving $e^{-\frac{i}{2k}}$ give no contribution to the linear integral equation (1.7) (see the discussion in Section 3). It will be shown below that, if $\beta < 0$, then $\rho_1(k) \neq 0$ for $\operatorname{Im} k \geq 0$. Thus, the functions M_1 and M_2 satisfy equation (1.17) where $\rho_2/\rho_1 = i\bar{\gamma}/(1-\beta)$. Letting $k = \frac{1}{2}\sqrt{\frac{\tau}{\chi}}\lambda$, $\xi = \sqrt{\tau\chi}$, equation (1.17) becomes

$$\begin{pmatrix} -N_2(\xi,\lambda) \\ N_1(\xi,\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\pi}\frac{\bar{\gamma}}{1-\beta}\int_{-\infty}^{\infty} e^{i\xi\left(\lambda'+\frac{1}{\lambda'}\right)} \begin{pmatrix} \bar{N}_1(\xi,\lambda') \\ \bar{N}_2(\xi,\lambda') \end{pmatrix} \frac{d\lambda'}{\lambda'-(\lambda-i0)}. \tag{4.5}$$

Noting that in this case

$$\frac{\bar{Y}_0(0)}{1-b_0(0)} = \frac{\bar{\gamma}}{1-\beta},$$

and comparing equation (4.5) with equation (1.20), it follows that *in this particular case, the general solution is given by the similarity solution, (1.18)–(1.20), where $Y_0(0)$ and $b_0(0)$ are replaced by γ and β, respectively.*

We now investigate the zeros of $\rho_1(k)$. Using equation (4.3), it follows that $\rho_1(k) = 0$ implies

$$2k = \frac{(2n-1)\pi - i\ln\left(\frac{1-\beta}{1+\beta}\right)}{[(2n-1)\pi]^2 + \left[\ln\left(\frac{1-\beta}{1+\beta}\right)\right]^2}, \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{4.6}$$

Thus, $\rho_1(k)$ has zeros with $\operatorname{Im} k > 0$ iff $(1-\beta)/(1+\beta) < 0$, i.e., iff $\beta > 0$. It is interesting to note that in this case there exist infinitely many such zeros. This property was first discussed in [10] (see also [5]).

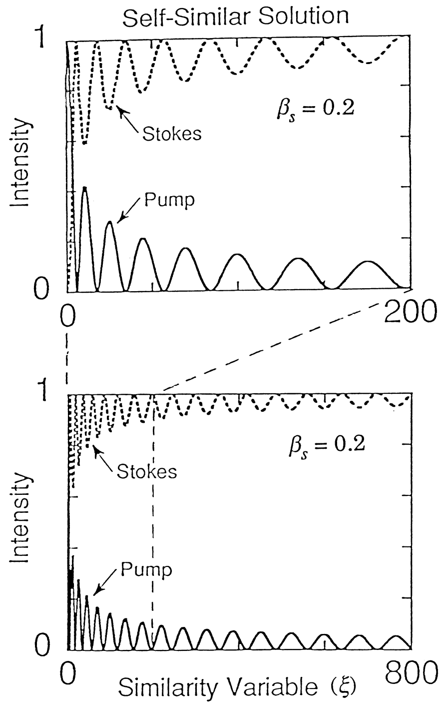

Fig. 2. A similarity solution. This type of solution is often called an accordion solution. The squeezing of the accordion from the right as ξ increases is visible. (Reprinted from [1].)

5. Numerical and Experimental Results

For completeness, we now discuss numerical solutions of equation (1.1). We also discuss the experiments performed to date as well as describe an experiment that could verify the predicted analytical behavior. A more elaborate discussion of these issues is given in [1] and [5].

In the examples presented here we assume that

$$b = \cos[\beta(\chi, \tau)], \qquad Y = i \sin[\beta(\chi, \tau)], \tag{5.1}$$

and thus it is sufficient to study the evolution of $\beta(\chi, \tau)$.[2] In the first example, we let $\beta(0, \tau) = 0.2$ (see Figure 2). In this case, as discussed in Section 4, the exact solution is given precisely by the similarity solution characterized by Painlevé III. Figure 2 depicts the numerical evaluation of $|A_1(\xi)|^2 = \cos^2[(\beta(\xi))]$ and of $|A_2(\xi)|^2 = \sin^2[\beta(\xi)]$, which are designated as the "pump" and the "Stokes," respectively. The self-similar

[2] In general, $b = \cos\beta$, $Y = e^{-i\theta}\sin\beta$; equations (5.1) correspond to $\theta(0, \tau) = 0$, which then implies $\theta(\chi, \tau) = 0$. An example where $\theta(0, \tau) \neq 0$ is discussed in [5].

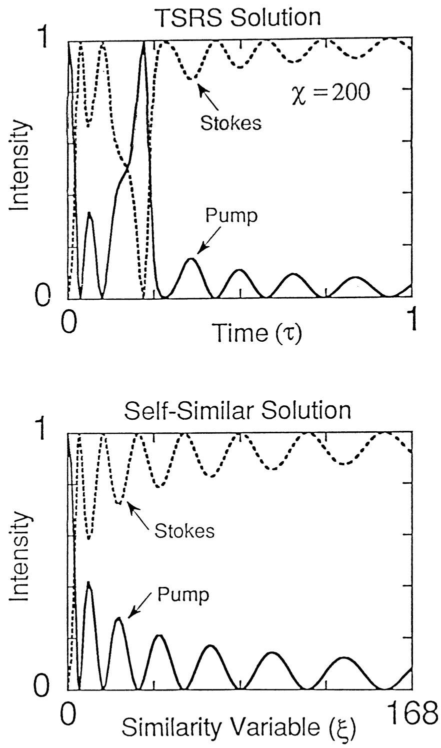

Fig. 3. The solution of the transient stimulated Raman scattering equations (1.1) are compared to the corresponding similarity solution at $\chi = 200$. The soliton leads to significant differences. (Reprinted from [1].)

nature of the solution is readily apparent. Actually, these solutions are often referred to in the physics literature as "accordions," because like an accordion squeezed from the right, the interesting ripples of the solutions are squeezed toward $\tau = 0$. In the second example, we let $\beta(0, \tau) = 0.2 - 2\pi\tau$. Figures 3 and 4 compare the numerical solution of the full PDEs (1.1) (labeled TSRS for transient stimulated Raman scattering) with the numerical solution of the similarity ODE setting $\beta(\xi = 0) = \beta_s = 0.2$, where $\xi = \sqrt{\chi\tau}$ is the similarity variable. It was shown in [5] that this comparison is best when the offset is given by

$$\chi_{\text{off}} = \frac{1}{\sin\beta_0(\tau)} \left. \frac{d}{d\tau}\beta_0(\tau) \right|_{\tau=0}, \qquad \beta_0(\tau) = \beta(0, \tau),$$

which, in this case, is approximately -32. This offset can be inferred from equations (1.18). For this reason the solution of the full TSRS equations (1.1) at $\chi = 200$ and $\chi = 800$ are compared with the similarity solutions in the range [0, 168] and the range [0, 768], respectively. A soliton is generated initially at $\tau = 0.2/2\pi$, due to the fact that there is a phase flip at this value of τ. This soliton propagates toward the right, and by

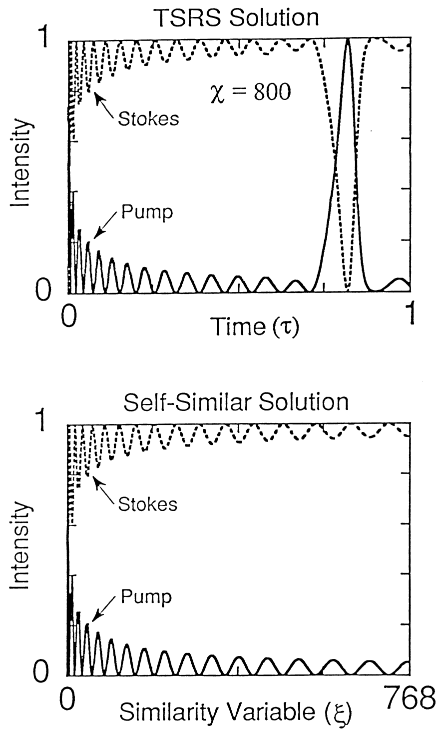

Fig. 4. The same comparison as in Fig. 3 is made at $\chi = 800$. The soliton has propagated to the back of the pulses and the agreement at times preceding the soliton is excellent. (Reprinted from [1].)

the point $\chi = 1000$, has "dropped off the edge." Beyond $\chi \sim 400$, there is excellent agreement between the TSRS solution at times preceeding the soliton and the similarity solution. Beyond $\chi = 1000$, there is an excellent agreement at all times.

In order to observe experimentally the predicted behavior in gases (such as H_2 or D_2), one must generate pulses that are short compared to the molecular de-excitation time or that have a rapid initial rise [1]. Indeed, self-similar oscillations have already been observed in the experiments of Duncan et al. [16] where 40ps pulses were used. In order to carry out a careful comparison of theory and experiments, one must use a multipass cell, like the one described by MacPherson et al. [2]. This type of cell filters out the higher-order Stokes and anti-Stokes radiation and also corresponds to a long length (for this, one needs 10 to 20 passes through the cell [1], [5]).

We conclude this section with two remarks.

1. The SRS system reduces to the sine-Gordon equation in the case that the optical fields are in phase. However, even in this case, the SRS system behaves very differently from the solutions of the Cauchy problem on the infinite line for the sine-Gordon equation.

The mathematical difference that reflects the physical difference is in the initial and boundary conditions. Because of these differences, the asymptotic behavior of the SRS system is characterized by the similarity solution.

2. If one wants to compare the solution of the SRS system with some particular solution of Painlevé III, one must choose some initial conditions for the Painlevé III equation (see the discussion above about the offset). An important advantage of the analysis presented here is that we have characterized *uniquely* the particular solution of the Painlevé III corresponding to any initial-boundary conditions of the SRS system. Indeed, equation (1.20) uniquely specifies the corresponding monodromy data that in principle characterizes the associated initial data (see [4]).

Appendix A. Small k Behavior of $\Phi(\chi, \tau, k)$

Substituting equations (3.11) and (3.12) into equation (3.1), we find that the small k behavior of the (11) and (21) terms of the first integral in equation (3.1) are given by

$$\begin{aligned} &iY(0,1)\left[\frac{\alpha_2(0,1)}{1+b(0,1)} - \frac{\beta_2(0,1)}{1-b(0,1)}\right]E(\tau,k) \\ &\qquad + \frac{iY(0,\tau)\beta_2(0,\tau)}{1-b(0,\tau)}e^{\frac{i}{2k}(\tau-1)} - \frac{iY(0,\tau)\alpha_2(0,\tau)}{1+b(0,\tau)}, \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} &i\bar{Y}(0,1)\left[\frac{\beta_1(0,1)}{1+b(0,1)} - \frac{\alpha_1(0,1)}{1-b(0,1)}\right]E(\tau,k) \\ &\qquad - \frac{i\bar{Y}(0,\tau)\beta_1(0,\tau)}{1+b(0,\tau)}e^{\frac{i}{2k}(\tau-1)} + \frac{i\bar{Y}(0,\tau)\alpha_1(0,\tau)}{1-b(0,\tau)}, \end{aligned} \tag{A.2}$$

where $E(\tau', \tau, k) = \exp\{-\frac{i}{4k}[\int_\tau^{\tau'} b(0,\xi)\,d\xi + 1 - \tau]\}$. Indeed, the leading behavior of the (11) term is given by

$$\begin{aligned} &\frac{1}{4k}\int_\tau^1 Y(0,\tau')\alpha_2(0,\tau')e^{-\frac{i}{4k}\left[\int_\tau^{\tau'} b(0,\xi)\,d\xi + \tau' - \tau\right]}\,d\tau' \\ &\qquad + \frac{1}{4k}\int_\tau^1 Y(0,\tau')\beta_2(0,\tau')e^{-\frac{i}{4k}\left[\int_\tau^{\tau'} b(0,\xi)\,d\xi + 2 - \tau - \tau'\right]}\,d\tau'. \end{aligned}$$

Integration by parts yields

$$i\frac{Y(0,\tau')\alpha_2(0,\tau')}{1+b(0,\tau')}e^{-\frac{i}{4k}\left[\int_\tau^{\tau'} b(0,\xi)\,d\xi+\tau'-\tau\right]}\Bigg|_\tau^1 + \frac{iY(0,\tau')\beta_2(0,\tau')}{-1+b(0,\tau')}e^{-\frac{i}{4k}\left[\int_\tau^{\tau'} b(0,\xi)\,d\xi+2-\tau-\tau'\right]}\Bigg|_\tau^1,$$

which upon simplification becomes (A.1).

Using (A.1), it follows that the leading order behavior of the (11) term of equation (3.1) implies

$$\begin{aligned}\alpha_1(\chi,\tau)+\beta_1(\chi,\tau)e^{\frac{i}{2k}(\tau-1)} &= E(1,\tau,k)\\ &\quad + iY(0,1)\left[\frac{\alpha_2(0,1)}{1+b(0,1)}-\frac{\beta_2(0,1)}{1-b(0,1)}\right]E(1,\tau,k)\\ &\quad + \frac{iY(0,\tau)\beta_2(0,\tau)}{1-b(0,\tau)}e^{\frac{i}{2k}(\tau-1)}-\frac{iY(0,\tau)\alpha_2(0,\tau)}{1+b(0,\tau)}\\ &\quad + \int_0^{\chi} X(\chi',\tau)\alpha_2(\chi',\tau)\,d\chi'\\ &\quad + \left(\int_0^{\chi} X(\chi',\tau)\beta_2(\chi',\tau)\,d\chi'\right)e^{\frac{i}{2k}(\tau-1)}.\end{aligned}$$

Thus,

$$\alpha_1(\chi,\tau)=-\frac{iY(0,\tau)\alpha_2(0,\tau)}{1+b(0,\tau)}+\int_0^{\chi} X(\chi',\tau)\alpha_2(\chi',\tau)\,d\chi', \tag{A.3a}$$

$$\beta_1(\chi,\tau)=\frac{iY(0,\tau)\beta_2(0,\tau)}{1-b(0,\tau)}+\int_0^{\chi} X(\chi',\tau)\beta_2(\chi',\tau)\,d\chi', \tag{A.3b}$$

$$1+iY(0,1)\left[\frac{\alpha_2(0,1)}{1+b(0,1)}-\frac{\beta_2(0,1)}{1-b(0,1)}\right]=0. \tag{A.3c}$$

Similarly, using (A.2), the (21) term of equation (3.1) implies

$$\alpha_2(\chi,\tau)=\frac{\bar{Y}(0,\tau)\alpha_1(0,\tau)}{1-b(0,\tau)}-\int_0^{\chi} \bar{X}(\chi',\tau)\alpha_1(\chi',\tau)\,d\chi', \tag{A.4a}$$

$$\beta_2(\chi,\tau)=-\frac{i\bar{Y}(0,\tau)\beta_1(0,\tau)}{1+b(0,\tau)}-\int_0^{\chi} \bar{X}(\chi',\tau)\alpha_2(\chi',\tau)\,d\chi', \tag{A.4b}$$

$$\frac{\beta_1(0,1)}{1+b(0,1)}=\frac{\alpha_1(0,1)}{1-b(0,1)}. \tag{A.4c}$$

Appendix B. Small k Behavior of the Spectral Data

Using an approach similar to the one used in Appendix A, it can be shown that the small k behavior of equation (3.20) yields

$$\Phi_1(0,\tau,k)=A_1(\tau)+B_1(\tau)e^{\frac{i}{2k}(\tau-1)}, \tag{B.1}$$

$$\Phi_2(0,\tau,k)=\frac{i\bar{Y}(0,\tau)}{1-b(0,\tau)}A_1(\tau)-\frac{i\bar{Y}(0,\tau)}{1+b(0,\tau)}B_1(\tau)e^{\frac{i}{2k}(\tau-1)}, \tag{B.2}$$

with

$$A_1(1) = \frac{1 - b(0,1)}{2}, \qquad B_1(1) = \frac{1 + b(0,1)}{2}. \tag{B.3}$$

Similarly, the small k behavior of equation (3.24) yields

$$\Phi_1(\chi, 0, k) = \alpha_1(\chi) + \beta_1(\chi)e^{-\frac{i}{2k}}, \tag{B.4}$$

$$\Phi_2(\chi, 0, k) = \alpha_2(\chi) + \beta_2(\chi)e^{-\frac{i}{2k}}, \tag{B.5}$$

where

$$\alpha_1(\chi) = A_1(0) + \int_0^\chi X(\chi', 0)\alpha_2(\chi')\, d\chi', \tag{B.6a}$$

$$\alpha_2(\chi) = \frac{i\bar{Y}(0,0)}{1 - b(0,0)} A_1(0) - \int_0^\chi \bar{X}(\chi', 0)\alpha_1(\chi')\, d\chi', \tag{B.6b}$$

$$\beta_1(\chi) = B_1(0) + \int_0^\chi X(\chi', 0)\beta_2(\chi')\, d\chi', \tag{B.7a}$$

$$\beta_2(\chi) = \frac{-i\bar{Y}(0,0)}{1 + b(0,0)} B_1(0) - \int_0^\chi \bar{X}(\chi', 0)\beta_1(\chi')\, d\chi'. \tag{B.7b}$$

Equations (3.28) imply

$$\rho_1(k) \sim \alpha_1(l) + \beta_1(l)e^{-\frac{i}{2k}}, \qquad k \to 0, \tag{B.8}$$

$$\rho_2(k) \sim \alpha_2(l) + \beta_2(l)e^{-\frac{i}{2k}}, \qquad k \to 0. \tag{B.9}$$

Thus, if $X(\chi, 0) = 0$,

$$\rho_1(k) \sim A_1(0) + B_1(0)e^{-\frac{i}{2k}}, \qquad k \to 0 \tag{B.10}$$

$$\rho_2(k) \sim \frac{i\bar{Y}(0,0)}{1 - b(0,0)} A_1(0) - \frac{i\bar{Y}(0,0)}{1 + b(0,0)} B_1(0)e^{-\frac{i}{2k}}, \qquad k \to 0. \tag{B.11}$$

Appendix C. The Dressing Method

We will show that it is possible to define $b(\chi, \tau)$, $Y(\chi, \tau)$, and $X(\chi, \tau)$ in terms of the asymptotic properties of $\Phi(\chi, \tau, k)$ in such a way that: (a) The $O(k)$, $O(1)$, and $O(\frac{1}{k})$ terms of the lhs of equations (3.18) vanish. (b) The functions b, Y, X satisfy equations (1.7).

We assume that $\Phi_{22} = \bar{\Phi}_{11}$, $\Phi_{12} = -\bar{\Phi}_{11}$. Substituting

$$\Phi = I + \frac{\Phi^{(1)}}{k} + \frac{\Phi^{(2)}}{k^2} + O\left(\frac{1}{k^3}\right), \qquad k \to \infty, \qquad k_I \neq 0, \tag{C.1}$$

in equations (3.18) we find

$$U = i[\sigma_3, \Phi^{(1)}], \tag{C.2}$$

$$\Phi^{(1)}_\chi + i[\sigma_3, \Phi^{(2)}] = U\Phi^{(1)}, \tag{C.3}$$

$$\Phi^{(2)}_\chi + i[\sigma_3, \Phi^{(3)}] = U\Phi^{(2)}, \tag{C.4}$$

$$\vdots$$

and

$$\Phi^{(1)}_\tau - i(1+b)\sigma_3 = -V, \tag{C.5}$$

$$\Phi^{(2)}_\tau - i(\Phi^{(1)}\sigma_3 + b\sigma_3\Phi^{(1)}) = -V\Phi^{(1)}, \tag{C.6}$$

$$\Phi^{(3)}_\tau - i(\Phi^{(2)}\sigma_3 + b\sigma_3\Phi^{(2)}) = -V\Phi^{(2)}, \tag{C.7}$$

$$\vdots$$

Equation (C.2) yields

$$U = \begin{pmatrix} 0 & X \\ -\bar{X} & 0 \end{pmatrix}, \qquad X = 2i\Phi^{(1)}_{12}. \tag{C.8}$$

Equation (C.3) yields

$$\Phi^{(1)}_{11_\chi} = \frac{|X|^2}{2i}, \tag{C.9}$$

$$X_\chi = 4\Phi^{(2)}_{12} - 2i\Phi^{(1)}_{11}X. \tag{C.10}$$

The diagonal part of (C.4) implies

$$\Phi^{(2)}_{11_\chi} = -X\bar{\Phi}^{(2)}_{12}. \tag{C.11}$$

Equation (C.5) yields

$$\Phi^{(1)}_{11_\tau} = i(1+b), \tag{C.12}$$

$$X_\tau = -2iY. \tag{C.13}$$

Equation (C.13) is equation (1.7c). Also, the compatibility of equations (C.9) and (C.12) implies equation (1.7a).

Equation (C.6) yields

$$\Phi^{(2)}_{11_\tau} = i(1+b)\Phi^{(1)}_{11} - \frac{Y\bar{X}}{2i}, \tag{C.14}$$

$$\Phi^{(1)}_{12_\tau} = \frac{(-1+b)}{2}X + Y\Phi^{(1)}_{11}. \tag{C.15}$$

The compatibility of the equations for $\Phi^{(2)}_{11_\chi}$ (equation (C.11)) and for $\Phi^{(2)}_{11_\tau}$ (equation (C.14)) imply equation (1.7b). Similarly the compatibility of the equations for X_χ (equation (C.10)) and for X_τ (equation (C.13)) also imply equation (1.7b).

In summary: Let the complex-valued function Φ satisfy $\Phi_{22} = \bar{\Phi}_{11}$, $\Phi_{12} = -\bar{\Phi}_{11}$. Define X, b, Y by

$$X = 2i\Phi^{(1)}_{12}, \qquad b = -1 - i\Phi^{(1)}_{11_\tau}, \qquad Y = \frac{i}{2}X_\tau. \tag{C.16}$$

Then if Φ satisfies (C.1), the $O(k)$, $O(1)$ and $O(\frac{1}{k})$ terms of equations (3.18) vanish iff X, b, Y satisfy equations (1.7).

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References

[1] C. R. Menyuk, Self-similarity in stimulated Raman scattering: An overview, in *Self-Similarity in Stimulated Raman Scattering*, ed. D. Levi, C. R. Menyuk, and P. Winternitz, Les Publications, CRM, Montreal, Canada, 1994.
[2] D. C. MacPherson, R. C. Swanson, and J. L. Carlsten, *IEEE J. Quantum Electron.* **QE-25**, 1741 (1989).
[3] A. S. Fokas, A unified transform method for solving linear and certain nonlinear PDE's, *Proc. Roy. Soc. Series A*, *Proc. Roy. Soc. Lond. A* **453**, 1411–1443 (1997).
[4] A. S. Fokas, U. Mugan, and X. Zhou, *Inverse Problems* **8**, 757–785 (1992).
[5] C. R. Menyuk, *Phys. Rev. A*, **47**, 2235 (1993).
[6] M. J. Ablowitz and A. S. Fokas, *Complex Variables and Applications*, Cambridge University Press, Cambridge, UK, 1997.
[7] P. Deift and X. Zhou, Announc., *Bull. Am. Math. Soc.*, **26**, 119 (1992).
[8] F. Y. F. Chu and A. C. Scott, *Phys. Rev. A* **12**, 2060 (1975).
[9] D. J. Kaup, *Physica* **6D**, 143 (1983); *Physica* **19D**, 125 (1986).
[10] D. J. Kaup and C. R. Menyuk, *Phys. Rev. A*, **42**, 1712, (1990).
[11] J. N. Elgin and T. B. O'Hare, *J. Phys. B*, **12**, 159 (1979).
[12] D. Levi, C. R. Menyuk, and P. Winternitz, *Phys. Rev. A* **44**, 6057 (1991).
[13] C. R. Menyuk and T. Seidman, *SIAM J. Math. Anal.* **23**, 346 (1992).
[14] A. S. Fokas and I. M. Gel'fand, *Lett. Math. Phys.* **32**, 189 (1994).
[15] A. S. Fokas and L. Y. Sung, *Inverse Problems*, **8**, 673 (1991).
[16] M. D. Duncan, R. Mahon, L. L. Tankersley, and J. Reintjes, *J. Opt. Soc. Am.* **B5**, 37–52 (1988).
[17] I. R. Gabitov, V. E. Zakharov, and A. V. Mikhailov, *Theor. Math. Phys.* **63**, 328 (1985); with J. Leon, *Phys. Rev. A* **47**, 3264 (1993).
[18] C. Claude and J. Leon, *Phys. Rev. Lett.* **74**, 3479 (1995); C. Claude, F. Ginovart and J. Leon, *Phys. Rev. A* **52**, 767 (1995).
[19] K. Drühl, R. G. Wenzel and J. L. Carlsten, *Phys. Rev. Lett.* **51**, 1171 (1983).