A reduced-complexity quadratic structure for the detection of stochastic signals

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Quadratic detection of discrete-time stochastic signals in additive stationary Gaussian noise is considered. A banded-quadratic detector structure is introduced to reduce the multiplicative complexity and data storage requirements of the optimum full-quadratic detector, and the optimization of this reduced-complexity structure is studied. The issue of performance versus complexity is explored for the specific problems of detecting wide-sense Markov and triangularly correlated signals in white noise, with the conclusion that performance of the reduced complexity detector can be very close to optimum if an adequate quadratic-form bandwidth is chosen.

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INTRODUCTION

In this paper we consider the problem of detecting the presence of stationary stochastic signals in additive stationary Gaussian noise in discrete time. A commonly used detection strategy for this situation consists of the comparison to a threshold of a quadratic form in the observations, and the optimization of this type of detector has been considered by a number of investigators (see, for example, Baker[1] or Gardner[2]). In the general case, for a detector using n observations, the computation of this optimum quadratic detection statistic requires $O(n^2)$ multiplications and the storage of up to $(n-1)$ past samples or transformed samples. Of course, for special signal and noise models, these figures may be reduced by recursive computations[3]; however, it is of interest to consider ways of reducing this complexity in the general case.

We address this problem by considering a class of quadratic detectors based on the comparison to a threshold of a banded-quadratic detection statistic. We investigate the asymptotic ($n \to \infty$) performance of these detectors and derive the optimum detector within this class. The particular case of detection in white noise is treated numerically for some specific signal models, with the conclusion that such detectors can perform very nearly optimally when designed properly, but that care must be exercised when choosing the width of the band in the banded-quadratic statistic.

The paper is organized as follows. In Sec. I some relevant preliminary results are presented. In Sec. II, an expression for the asymptotic performance of banded-quadratic detectors is derived, and the optimization of the banded detectors is considered. Section III contains the treatment of the white-noise case, and Sec. IV contains some concluding remarks.

I. PRELIMINARIES

Consider the following hypothesis-testing model for the problem of detecting a stochastic signal in additive noise:

$$H_0:\quad X_i = N_i,\quad i=1,2,\ldots,n$$

vs (1)

$$H_1:\quad X_i = N_i + \theta S_i,\quad i=1,2,\ldots,n,$$

where $\mathbf{X}_n = (X_1, X_2, \ldots, X_n)^T$ is a random observation vector, $\{N_1, N_2, \ldots\}$ is a zero-mean stationary Gaussian noise sequence, $\{S_1, S_2, \ldots\}$ is a stationary second-order (i.e., $E\{S_i^2\} < \infty$) signal sequence independent of the noise, and θ is a positive signal-to-noise ratio parameter. We assume that $\{N_1, N_2, \ldots\}$ has a power spectral density (PSD) f_N which is bounded away from 0 and ∞.

We will be considering detectors for (1) of the form

$$\varphi_n(\mathbf{x}_n) = \begin{cases} 1, & \text{if } T_n(\mathbf{x}_n) \geqslant \tau_n, \\ 0, & \text{if } T_n(\mathbf{x}_n) < \tau_n, \end{cases} \tag{2}$$

where $\mathbf{x}_n$ is the observed realization of $\mathbf{X}_n$, the value $\varphi(\mathbf{x}_n) = j$ indicates the choice of hypothesis H_j, T_n is a detection statistic (i.e., a real-valued function of the observed data), and τ_n is a decision threshold. A useful measure of detection performance for the detectors of the form of (2) in the detection model of (1) is the deflection[1,2] or generalized SNR,[4] defined by

$$S(T_n) = \frac{(E_1\{T_n(\mathbf{X}_n)\} - E_0\{T_n(\mathbf{X}_n)\})^2}{\mathrm{Var}_0(T_n(\mathbf{X}_n))}, \tag{3}$$

where $E_j\{\cdot\}$ denotes expectation under hypothesis $H_j (j=0,1)$ and $\mathrm{Var}_0(\cdot)$ denotes variance under hypothesis H_0. For the problem under consideration here, we are interested primarily in the large-sample-size (i.e., large n) situation since the detector complexity is most critical in this case. Thus for a sequence of detectors based on statistics $T_1, T_2, \ldots$ we define the *asymptotic deflection* by

$$S_a(T_\infty) = \lim_{n \to \infty} [S(T_n)/n], \tag{4}$$

and we will compare two detection schemes using the ratio of their asymptotic deflections [dividing by n in (4) keeps the limit finite and nonzero for the detectors of interest here]. Within mild regularity conditions, this ratio (as $\theta \to 0$) also gives the low-SNR Pitman asymptotic relative efficiency[5] of

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the two detection schemes, which is a measure of the relative number of samples the two schemes require to achieve identical performance. Using either interpretation, this criterion gives a good indication of large-sample detection performance.

Before considering the problem of designing quadratic detectors for (1), it is of interest to consider the situation of (1) in which the signal sequence is a known positive constant, i.e., $S_i = \mu > 0$ for all i. In this case a useful detection strategy for (1) is of the form of (2) with statistic

$$T_{l,n}(\mathbf{x}_n) = \mathbf{1}_n^T \Sigma_{N,n}^{-1} \mathbf{x}_n, \tag{5}$$

where $\mathbf{1}_n^T = (1,1,...,1)_{n\times 1}$ and where $\Sigma_{N,n}$ is the covariance matrix of $(N_1,...,N_n)^T$. This detection scheme is optimum for (1) under several criteria (by proper choice of threshold), including the Neyman–Pearson, Bayes, and maximum-deflection criteria (see, e.g., Ref. 6), and is locally ($\theta \to 0$) optimum for (1) even when the signal is random if $E\{S_i\} > 0$. [A discussion of the local optimality of (5) is included in the Appendix.] Another common detector for this situation is the simple linear detector based on the statistic

$$T_{sl,n}(\mathbf{x}_n) = \sum_{i=1}^{n} x_i = \mathbf{1}_n^T \mathbf{x}_n, \tag{6}$$

which, although not optimum for (1) unless the noise is white, is less complex than the optimum detector of (5). The asymptotic deflection ratio of (6) relative to (5) is given straightforwardly by

$$S_a(T_{sl,\infty})/S_a(T_{l,\infty}) = \lim_{n\to\infty} [(\mathbf{1}_n^T \Sigma_{N,n}^{-1} \mathbf{1}_n/n) \times (\mathbf{1}_n^T \Sigma_{N,n} \mathbf{1}_n/n)]^{-1}. \tag{7}$$

The limit of (7) equals unity under very mild conditions on the noise spectrum f_N (see Grenander[7] or Davisson[8]); thus, for the deterministic-signal version of (1), the simple detector of (6) is asymptotically equivalent to the optimum detector of (5). This equivalence is not too surprising if one notes that, within mild regularity, $\Sigma_{N,n}^{-1}$ is asymptotically ($n \to \infty$) Toeplitz; i.e., $\lim_{n\to\infty} \Sigma_{N,n}^{-1}$ has elements $a_{ij} = a_{|i-j|}$. This implies that

$$\mathbf{1}_n^T \Sigma_{N,n}^{-1} \mathbf{x}_n \sim \left(1 + 2\sum_{k=0}^{\infty} a_0\right) \mathbf{1}_n^T \mathbf{x}_n,$$

which is equivalent to (6) since the constant term can be absorbed into the threshold.

Now, and for the remainder of this paper, we consider the situation of (1) in which the signal is random with zero mean. In this case it is conventional to use a quadratic detector for (1), and the optimum (maximum-deflection) quadratic detector uses the statistic[1]

$$T_{q,n}(\mathbf{x}_n) = \mathbf{x}_n^T W_n \mathbf{x}_n, \tag{8}$$

where

$$W_n = \Sigma_{N,n}^{-1} \Sigma_{S,n} \Sigma_{N,n}^{-1}, \tag{9}$$

and $\Sigma_{S,n}$ is the covariance matrix of $(S_1,S_2,...,S_n)^T$. The quadratic detector of (8) is also locally optimum for the zero-mean signal case of (1) (see the Appendix). An alternative to (8) for detecting random signals is the simple quadratic detector or energy detector which uses the statistic

$$T_{sq,n}(\mathbf{x}_n) = \sum_{i=1}^{n} x_i^2 = \mathbf{x}_n^T \mathbf{x}_n. \tag{10}$$

Asymptotically ($n \to \infty$) and locally ($\theta \to \infty$), the simple quadratic detector is the optimum detection statistic for (1) of the form $\Sigma_{i=1}^n g(x_i)$, where g is any memoryless nonlinearity. This is shown in Ref. 9 for the case in which the noise is m-dependent (i.e., $E\{N_i N_{i+k}\} = 0$ for $|k| > m$) and follows straightforwardly from results in Ref. 10 by an argument similar to that in Ref. 9 for the general case. Unfortunately, as we will see below, the simple quadratic detector is not generally asymptotically equivalent to (8) and can exhibit quite poor performance relative to (8). For example, suppose the noise is white so that $\Sigma_{N,n} = \sigma_0^2 I_n$ for some $\sigma_0^2 > 0$, where I_n is the $n \times n$ identity matrix. In this case, it follows from the results of Sec. II below, that the asymptotic deflection ratio of (10) relative to (8) is given by

$$\frac{S_a(T_{sq,\infty})}{S_a(T_{q,\infty})} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f_S(\omega)d\omega\right)^2 \Big/ \frac{1}{2\pi}\int_{-\pi}^{\pi} f_S^2(\omega)d\omega, \tag{11}$$

where f_S is the PSD of the signal sequence. It follows from the Schwarz inequality that the quantity of (11) is strictly less than unity unless f_S is constant on $[-\pi,\pi]$ (i.e., unless the signal is white).

II. PERFORMANCE OPTIMIZATION OF BANDED-QUADRATIC DETECTORS

In general, the computation of the quadratic statistic of (8) requires $O(n^2)$ multiplications and the storage of $(n-1)$ data samples or transformed data samples. Alternatively, the simple quadratic detector of (10) requires only n multiplications and the storage of only the accumulated sum of squares. However, as noted above, this simplified detector can perform poorly as compared to (8), and thus it is of interest to consider a class of quadratic detectors with complexities intermediate to these two extremes in hopes of achieving near-optimum performance with lower complexity than that required for (8). One such class of detectors is the class of banded-quadratic detectors of the form of (2) with statistic

$$T_{M,n}(\mathbf{x}_n) = \mathbf{x}_n^T B_{M,n} \mathbf{x}_n, \tag{12}$$

where $B_{M,n}$ is a banded $n \times n$ Toeplitz matrix with bandwidth $(2M+1)$; i.e., $B_{M,n}$ has elements $b_{i,j}$ satisfying $b_{i,j} = b_{|i-j|}$ and $b_{|i-j|} = 0$ for $|i-j| > M$. In this section we consider the performance optimization of detectors of the form of (12).

The deflection of the quadratic form of (12) is given by (see, e.g., Ref. 1)

$$S(T_{M,n}) = \frac{\theta^4 (\mathrm{tr}\{B_{M,n}\Sigma_{S,n}\})^2}{2\,\mathrm{tr}\{B_{M,n}\Sigma_{N,n}B_{M,n}\Sigma_{N,n}\}}. \tag{13}$$

Since the trace of a matrix is the sum of its eigenvalues we can write (13) as

$$S(T_{M,n}) = \theta^4 \left(\sum_{i=1}^{n} \lambda_{S,i}^{(n)}\right)^2 \Big/ 2\sum_{i=1}^{n} (\lambda_{N,i}^{(n)})^2, \tag{14}$$

where $\{\lambda_{S,1}^{(n)}, \lambda_{S,2}^{(n)},...,\lambda_{S,n}^{(n)}\}$ and $\{\lambda_{N,1}^{(n)}, \lambda_{N,2}^{(n)},...,\lambda_{N,n}^{(n)}\}$ are the eigenvalues of $B_{M,n}\Sigma_{S,n}$ and $B_{M,n}\Sigma_{N,n}$, respectively. Because of the Toeplitz structure of $B_{M,n}, \Sigma_{S,n}$, and $\Sigma_{N,n}$, it follows that (see Grenander and Szegö[11])

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\lambda_{S,i}^{(n)}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f_S(\omega)f_M(\omega)d\omega, \qquad (15)$$

where f_S is the spectrum of $\{S_1,S_2,...\}$ and where f_M is the Fourier series associated with the sequence $\{b_0,b_{\pm 1},...,b_{\pm M}\}$; i.e.,

$$f_M(\omega)=b_0+2\sum_{j=1}^{M}b_j\cos(j\omega),\quad -\pi\leqslant\omega\leqslant\pi. \qquad (16)$$

Similarly, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(\lambda_{N,i}^{(n)})^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}f_N^2(\omega)f_M^2(\omega)d\omega, \qquad (17)$$

and thus

$$S_a(T_{M,\infty})=\theta^4\left(\frac{1}{2\pi}\int_{\pi}^{\pi}f_S(\omega)f_M(\omega)d\omega\right)^2\bigg/\frac{1}{\pi}\int_{\pi}^{\pi}f_N^2(\omega)f_M^2(\omega)d\omega. \qquad (18)$$

Since f_M is a trigonometric polynomial, the numerator term of (18) can be written as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}f_S(\omega)f_M(\omega)d\omega=\mathbf{r}_M^T\mathbf{b}_M \qquad (19)$$

when $\mathbf{b}_M=(b_0,b_1,...,b_M)^T$ and $\mathbf{r}_M=(r_0,2r_1,...,2r_M)^T$ with $r_j=E\{S_i S_{i+j}\}$, $j=0,1,...,M$. Similarly, since $f_M(\omega)=\mathbf{b}_M^T\mathbf{U}_M(\omega)$, where $\mathbf{U}_M(\omega)=[1,2\sin(\omega),...,2\sin(M\omega)]^T$, we can write the denominator of (18) as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}f_N^2(\omega)f_M^2(\omega)d\omega=\mathbf{b}_M^TC_m\mathbf{b}_M, \qquad (20)$$

where the matrix C_M is given by

$$C_M=\frac{1}{2\pi}\int_{-\pi}^{\pi}f_N^2(\omega)\mathbf{U}_M(\omega)\mathbf{U}_M^T(\omega)d\omega. \qquad (21)$$

Thus we have

$$S_a(T_{M,\infty})=\theta^4(\mathbf{r}_M^T\mathbf{b}_M)^2/2\mathbf{b}_M^TC_M\mathbf{b}_M. \qquad (22)$$

Note that, since we have assumed $f_N(\omega)>0$, the matrix C_M is positive definite and hence is invertible. It follows from (22) and the Schwarz inequality that the maximum value of $S_a(T_{M,\infty})$ is achieved by the coefficient vector

$$\mathbf{b}_{M,\mathrm{opt}}=C_M^{-1}\mathbf{r}_M, \qquad (23)$$

and this maximum value is

$$\max_{\mathbf{b}_M}S_a(T_{M,\infty})=(\theta^4/2)\mathbf{r}_M^TC_M^{-1}\mathbf{r}_M. \qquad (24)$$

Equation (23) specifies the optimum quadratic form for the statistic of (12), and (24) specifies its performance. The matrix C_M appearing in these expressions can be written in simpler form if we note that

$$f_N^2(\omega)=\gamma_0+2\sum_{k=1}^{\infty}\gamma_k\cos(k\omega),\quad -\pi\leqslant\omega\leqslant\pi, \qquad (25)$$

where

$$\gamma_k=\sum_{j=-\infty}^{\infty}\rho_j\rho_{j+k} \qquad (26)$$

with $\rho_j=E\{N_iN_{i+j}\}, j=0,\pm 1,\pm 2,...$. Assuming that order of integration and summation can be interchanged in (21) (a sufficient condition for this interchange is $\Sigma_{k=1}^{\infty}|\gamma_k|<\infty$), the elements of C_M are given straightforwardly by

$$(C_M)_{i,j}=\begin{cases}\gamma_0, & \text{if } i=j=0,\\ 2\gamma_{i+j}, & \text{if } i=0 \text{ or } j=0 \text{ but } i\neq j,\\ 2(\gamma_{i+j}+\gamma_{|i-j|}), & \text{if } i>0 \text{ and } j>0.\end{cases} \qquad (27)$$

Thus the determination of the optimum banded-quadratic structure requires only the computation of the inverse of an easily obtained $(M+1)\times(M+1)$ matrix.

III. RESULTS FOR DETECTION IN WHITE NOISE

Of particular interest in applications is the model of (1) in which the noise is white, i.e., in which the noise covariance matrix is given for each n by $\Sigma_{N,n}=\sigma_0^2I_n$, where $\sigma_0^2>0$ and I_n is the $n\times n$ identity matrix. In this section, we consider this case in some detail and explore the tradeoff between the number of quadratic terms in (12) and the detection performance.

In the white-noise case, we have $\rho_0=\sigma_0^2$ and $\rho_j=0$ if $j\neq 0$ where, as before, $\rho_j=E\{N_iN_{i+j}\}$. This implies that the sequence $\{\gamma_k\}$ defined by (26) is given by

$$\gamma_k=\begin{cases}\sigma_0^4, & k=0,\\ 0, & k\neq 0.\end{cases} \qquad (28)$$

Thus from (27), we have that C_M is a diagonal matrix,

$$C_M=\sigma_0^4\,\mathrm{diag}\,\{1,2,...,2\}, \qquad (29)$$

and $C_M^{-1}=\sigma_0^{-4}\,\mathrm{diag}\,\{1,\frac{1}{2},...,\frac{1}{2}\}$. From (23) we have that the optimum $(2M+1)$ banded-quadratic form is specified by

$$\mathbf{b}_{M,\mathrm{opt}}=\sigma_0^{-4}(r_0,r_1,...,r_M)^T, \qquad (30)$$

and from (24) we have that the optimum (banded) performance is given by

$$\max_{\mathbf{b}_M}S_a(T_{M,\infty})=\theta^4\left(r_0^2+2\sum_{k=1}^{M}r_k^2\right)\bigg/2\sigma_0^4. \qquad (31)$$

Note that the optimum quadratric form from (8) is given for this case by

$$T_{q,n}(\mathbf{x}_n)=\sigma_0^{-4}\mathbf{x}_n^T\Sigma_{S,n}\mathbf{x}_n. \qquad (32)$$

Thus, in this case, the optimum banded-quadratic form is simply a truncated version of the optimum unbanded one [recall that $r_k=(\Sigma_{S,n})_{i,i+k}$].

Since $\Sigma_{S,n}$ is a Toeplitz matrix, the asymptotic deflection of the optimum statistic of (32) is given straightforwardly by

$$S_a(T_{q,\infty})=\frac{\theta^4}{4\pi\sigma_0^4}\int_{\pi}^{\pi}f_S^2(\omega)d\omega=\theta^4\left(r_0^2+2\sum_{k=1}^{\infty}r_k^2\right)\bigg/2\sigma_0^2. \qquad (33)$$

Thus the asymptotic deflection ratio (ADR) of the optimum $(2M+1)$-banded-quadratic form relative to the optimum quadratic form is given by

$$\mathrm{ADR}_{M,\mathrm{opt}}=\left(r_0^2+2\sum_{k=1}^{M}r_k^2\right)\bigg/\left(r_0^2+2\sum_{k=1}^{\infty}r_k^2\right), \qquad (34)$$

which allows one to investigate the tradeoff between the number of quadratic terms (indicated by M) and performance. [As noted above, this quantity also indicates Pitman asymptotic relative efficiency under suitable regularity conditions. Most of the required regularity conditions follow straightforwardly for quadratic forms in Gaussian variables (e.g., asymptotic normality; Grenander and Szegö[11]).] To illustrate this tradeoff we consider the following two examples.

A. Example 1. Wide-sense Markov signal

We first consider the case in which the signal correlation structure is given by

$$r_k = Pr^{|k|}, \quad k = 0, \pm 1, \ldots, \tag{35}$$

where $P > 0$ is the signal energy and $|r| < 1$ is the signal correlation parameter. This corresponds to the case in which the signal is wide-sense Markov (strictly Markov if the signal is Gaussian), and the corresponding signal spectral density is given by the Poisson kernel

$$f_S(\omega) = \frac{P(1-r^2)}{1 - 2r\cos(\omega) + r^2}, \quad -\pi \leqslant \omega \leqslant \pi. \tag{36}$$

For $r > 0$ such a signal can arise, for example, from uniform sampling of a continuous-time signal with spectral density

$$\frac{2\alpha P}{\alpha^2 + \omega^2}, \quad -\infty < \omega \infty. \tag{37}$$

Here, $f_{3\text{ dB}} \triangleq \alpha/2\pi$ is the one-sided 3-dB power bandwidth (in Hz) of the unsampled signal, and r is given by

$$r = \exp\{-2\pi f_{3\text{dB}}/f_s\}, \tag{38}$$

where f_s is the sampling frequency.

In this case we have

$$\text{ADR}_{M,\text{opt}} = 1 - 2r^{2M+2}/(1 + r^2),$$

which is plotted versus $|r|$ for several values of M in Fig. 1. Note that the performance degradation due to truncation can be severe for this case if an adequately large M is not used, particularly for $|r|$ near 1. On the other hand, when an adequate M is chosen, the performance is very near optimum. [Note that, in this particular case, the optimum statistic of (32) can be computed via a recursion since the signal is of the autoregressive type. However, these results serve to illustrate the tradeoff of interest for signals with approximately exponentially decaying correlation.]

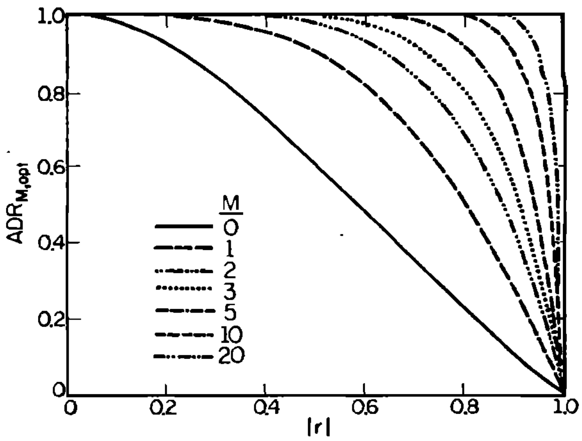

FIG. 1. Performance of the optimum $(2M+1)$-banded-quadratic detector relative to the optimum quadratic detector for a wide-sense Markov signal ($r_k = Pr^{|k|}$) in additive white noise.

B. Example 2. Triangularly correlated signal

As a second example we consider the case in which the signal correlation structure is given by

$$r_k = \begin{cases} P(1 - |k|/m), & |k| < m \\ 0, & |k| \geqslant m \end{cases} \tag{39}$$

where, as before, $P > 0$ is the signal energy and $m > 0$ is the correlation length of the signal. The signal spectral density corresponding to (38) is given by the mth Fejér kernel,

$$f_S(\omega) = \frac{P}{m}\left(\frac{\sin(m\omega/2)}{\sin(\omega/2)}\right)^2, \quad -\pi \leqslant \omega \leqslant \pi. \tag{40}$$

Such a signal can be obtained by uniform sampling of a continuous-time signal with spectrum

$$\frac{PT\sin^2(\omega T/2)}{(\omega T/2)^2}, \quad -\infty < \omega < \infty. \tag{41}$$

The parameter m is then given by $m = f_s T$ where f_s is the sampling frequency. The one-sided 3-dB power bandwidth, $f_{3\text{ dB}}$, of this spectrum is approximately $1.39/\pi T$ Hz, implying that m is given approximately by

$$m \cong 1.39 f_s/\pi f_{3\text{ dB}}. \tag{42}$$

The asymptotic deflection ratio of the optimum $(2M+1)$-banded-quadratic detector relative to the optimum quadratic detector is given straightforwardly by

$$\text{ADR}_{M,\text{opt}} = \begin{cases} \dfrac{1 + (2M^3 + 3M^2 + M - 6mM^2 - 6mM)/3m^2}{1 + (m-1)(2m-1)/3m}, & M < m, \\ 1, & M \geqslant m. \end{cases} \tag{43}$$

Note that the unity value for $M \geqslant m$ is due to the fact that $\Sigma_{S,n}$ is itself of banded form with bandwidth $(2m+1)$. Equation (43) is plotted versus m for several values of M in Fig. 2 and versus M for several values of m in Fig. 3. Note that the choice of M is again critical, and these figures indicate that a value of $M \cong m/2$ is adequate to achieve at least 90% of the optimum performance.

Note that the potential performance degradation here appears to be much worse than that in Example 1 for small values of M. However, the scales of Figs. 1 and 2 are not directly comparable on a linear basis. To compare these two figures, we can consider the signal sequences to be sampled versions of continuous-time wide-sense and triangularly correlated signals as discussed above. With this interpretation, if we equate sampling rates relative to 3-dB signal bandwidth [i.e., $f_s/f_{3\text{ dB}}$ in (38) and (42)], $m = 10$ in Fig. 2 is approxi-

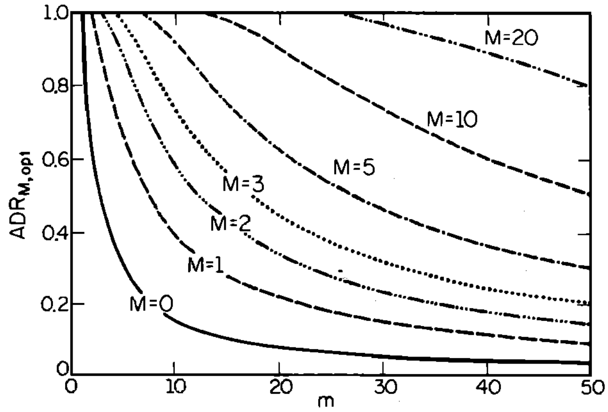

FIG. 2. Performance of the optimum $(2M+1)$-banded-quadratic detector relative to the optimum quadratic detector for a triangularly correlated signal $[r_k = P(1-|k|/m)$ if $|k|<m$ and $r_k = 0$ if $|k|>m]$ in additive white noise.

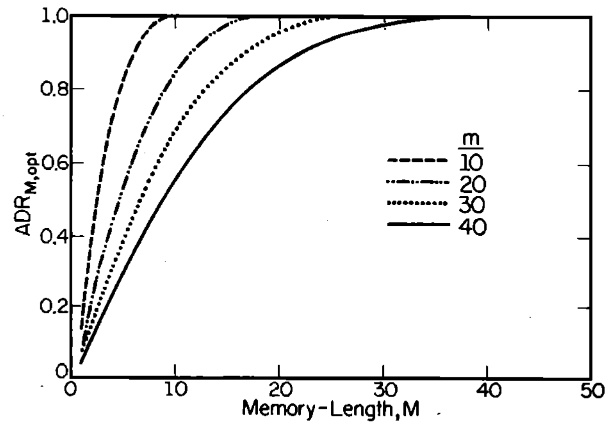

FIG. 3. Performance of the optimum $(2M+1)$-banded-quadratic detector relative to the optimum quadratic detector for a triangularly correlated signal $[r_k = P(1-|k|/m)$ if $|k|<m$ and $r_k = 0$ if $|k|>m]$ in additive white noise.

mately equivalent to $r=0.76$ in Fig. 1, $m=20$ to $r=0.87$, $m=30$ to $r=0.91$, $m=40$ to $r=0.93$, and $m=50$ to $r=0.95$. Examination of these results within this interpretation shows the two signal correlation structures to give comparable results.

IV. CONCLUSIONS

In this paper we have considered the design and performance of quadratic detectors with a limited number of quadratic terms. The optimization of such detectors was considered in Sec. II, where it was shown that the optimum coefficients are given by a linear transformation of the first M signal covariances, where $2M+1$ is the width of the banded-quadratic form describing the detection statistic. The white-noise case is particularly simple and, in Sec. III, this case was investigated in some detail with the conclusion that the number of quadratic terms is an important design consideration in quadratic detectors since poor performance can result from inadequate choice of this parameter.

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APPENDIX: LOCAL OPTIMALITY IN THE MODEL OF EQUATION (1)

For the model of (1), the likelihood ratio is given by (see, e.g., Van Trees[6])

$$L_\theta(\mathbf{x}_n) = \int_{\mathbf{R}^n} \exp\{\theta \mathbf{s}^T \Sigma_{N,n}^{-1} \mathbf{x}_n - \frac{\theta^2}{2} \mathbf{s}^T \Sigma_{N,n}^{-1} \mathbf{s}\} dG_n(\mathbf{s}), \quad \text{(A1)}$$

where G_n is the joint probability distribution of $\mathbf{S} = (S_1,\ldots,S_n)^T$. Assuming $E\{\mathbf{S}\} \neq 0$, the locally optimum Neyman–Pearson detection statistic (i.e., the detector which maximizes the slope of the detection probability at $\theta = 0$ for fixed false-alarm probability; see, for example, Capon[5] or Ferguson[12]) is given by

$$\frac{\partial}{\partial\theta} L_\theta(\mathbf{x}_n)|_{\theta=0} = \int_{\mathbf{R}^n} \mathbf{s}^T \Sigma_{N,n}^{-1} \mathbf{x}_n \, dG_n(\mathbf{s}) = \mu \mathbf{1}_n^T \Sigma_{N,n}^{-1} \mathbf{x}_n, \quad \text{(A2)}$$

where $E\{\mathbf{S}\} = \mu \mathbf{1}_n^T$ and we have assumed sufficient regularity on G_n to allow interchanging the order of integration and differentiation. Since the constant μ can be incorporated into the detection threshold, the statistic of (A2) is equivalent to that of (5). If, on the other hand, $E\{\mathbf{S}\} = \mathbf{0}$ then we seek a locally optimum unbiased detector (see Ferguson[12]) and this is given (again assuming sufficient regularity on G_n) by the statistic

$$\frac{\partial^2}{\partial\theta^2} L_\theta(\mathbf{x}_n)|_{\theta=0} = \int_{\mathbf{R}^n} [(\mathbf{s}^T \Sigma_{N,n}^{-1} \mathbf{x}_n)^2 - \mathbf{s}^T \Sigma_{N,n}^{-1} \mathbf{s}] dG_n(\mathbf{s})$$
$$= \mathbf{x}_n^T \Sigma_{N,n}^{-1} \Sigma_{S,n} \Sigma_{N,n}^{-1} \mathbf{x}_n - \mathrm{tr}\{\Sigma_{S,n} \Sigma_{N,n}^{-1}\}, \quad \text{(A3)}$$

where $\Sigma_{S,n} = E\{\mathbf{S}\mathbf{S}^T\} = \mathrm{cov}(\mathbf{S})$.

Since the term $\mathrm{tr}\{\Sigma_{S,n} \Sigma_{N,n}^{-1}\}$ in (A3) does not depend on the observations, it can be incorporated into the threshold; and thus (A3) is equivalent to (9). For related results concerning local optimality in the model of (1) see Refs. 13–16.

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