NAME: SOLUTIONS

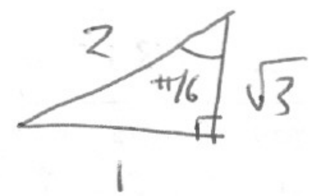
1 /15	2 /15	3 /18	4 /15	5 /12	T /75

MATH 251 (Spring 2008) Exam 1, Feb 20th

No calculators, books or notes! Show all work and give **complete explanations**.
This 65 minute exam is worth a total of 75 points.

(1) [15 pts] Let **a** and **b** be two vectors so that $|\mathbf{a}| = 2$, $|\mathbf{b}| = 5$ and the angle between **a** and **b** is $\pi/6$.
(a) Find $\mathbf{a} \cdot \mathbf{b}$.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = 2 \cdot 5 \cos \pi/6$$
$$= 10 \frac{\sqrt{3}}{2}$$
$$= 5\sqrt{3}$$



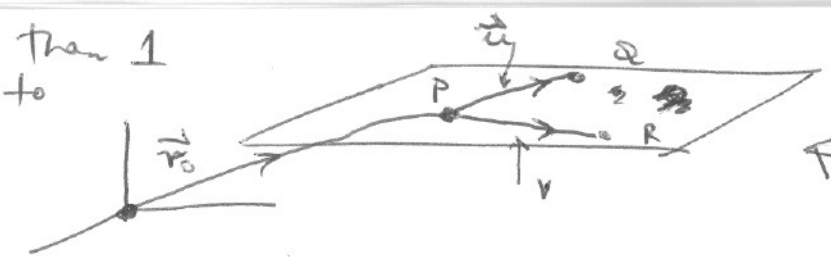
(b) Calculate the scalar projection of **b** onto **a**.

$$\mathrm{COMP}_{\vec{a}}(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{5\sqrt{3}}{2}$$ from (a)

(c) Find the area of the parallelogram determined by **a** and **b**.

$$\text{AREA} = |\vec{a} \times \vec{b}|$$
$$= |\vec{a}||\vec{b}| |\sin\theta|$$
$$= 2 \cdot 5 |\sin \pi/6|$$
$$= 10 \cdot \frac{1}{2} = 5.$$

There is more than 1 correct answer to (a) and (b)



(2) [15 pts]
(a) Find a *parametric equation* for the plane through the points $(3,-1,2)$, $(8,2,4)$, and $(-1,-2,-3)$.

$\vec{P} = (3,-1,2) \qquad \vec{Q} = (8,2,4), \qquad \vec{R} = (-1,-2,-3)$

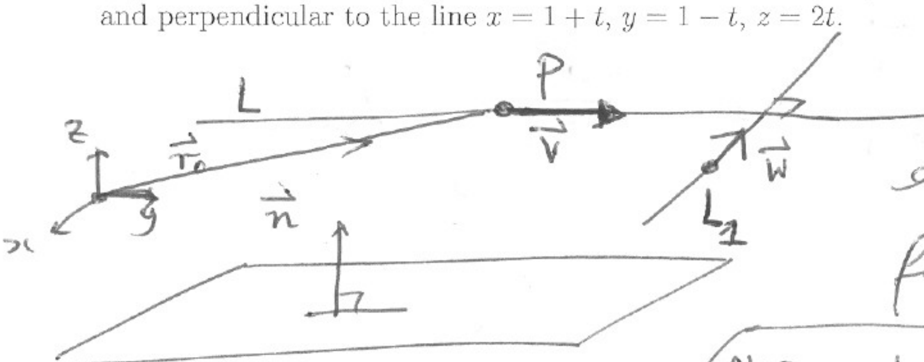
$\vec{u} = \vec{PQ} = \vec{Q} - \vec{P} = (5,3,2)$
$\vec{v} = \vec{PR} = \vec{R} - \vec{P} = (-4,-1,-5)$
These are 2 vectors in the plane

$\vec{r}_0 = \vec{P} = (3,-1,2)$ Vector whose endpoint is in the plane

$\vec{r}(s,t) = \vec{r}_0 + s\vec{u} + t\vec{v} = (3,-1,2) + s(5,3,2) + t(-4,-1,-5)$
$= (3+5s-4t,\ -1+3s-t,\ 2+2s-5t)$

MANY OF YOU FOUND NORMAL VECTOR $\vec{n} = \vec{u} \times \vec{v}$ and then got LEVEL SET EQN $(\vec{x} - \vec{x}_0) \cdot \vec{n} = 0$ where $\vec{x}_0 = \vec{P}$. This is not what I asked for.

(b) Find a *parametric equation* for the line through the point $(0,1,2)$ that is parallel to the plane $x+y+z=2$ and perpendicular to the line $x=1+t$, $y=1-t$, $z=2t$.



GOAL
To find parametric equation $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ for the line L.

NOTE L_1 IS NOT perpendicular to plane!

$\vec{r}_0 = P = (0,1,2)$

Since the line L is parallel to plane, the vector $\vec{v}$ is parallel to a vector that lies in the plane and so must be perpendicular to the normal vector $\vec{n} = (1,1,1)$ to the plane. Since L is perpendicular to L_1, $\vec{v}$ and $\vec{w}$ are perpendicular. So choose $\vec{v} = \vec{n} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix}$

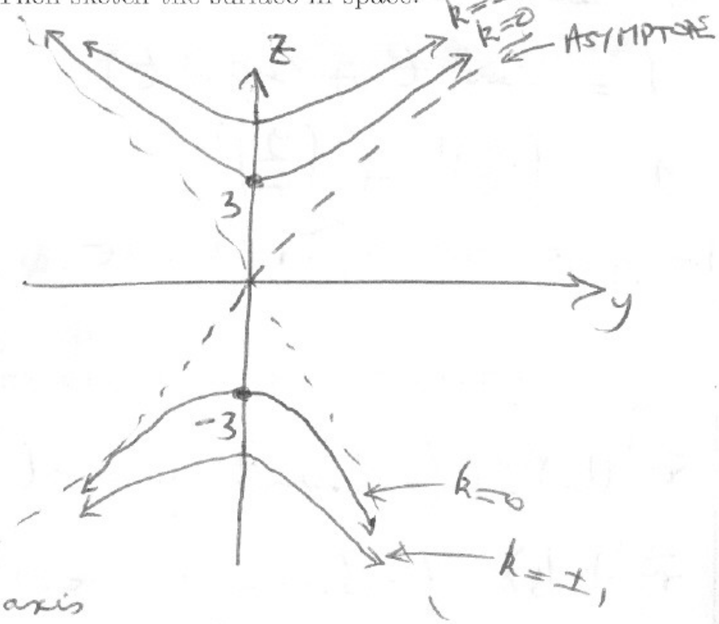
So $\vec{r}(t) = (0,1,2) + t(3,-1,-2)$.

(3) [18 pts] Consider the quadric surface

$$x^2 + \left(\frac{y}{2}\right)^2 - \left(\frac{z}{3}\right)^2 = -1.$$

Find equations for the traces of this surface in the planes $x = k$, $y = k$, and $z = k$ for a few appropriately chosen values of k. Sketch each of these traces in a plane. Then sketch the surface in space.

$\underline{x=k}$ $\left(\frac{z}{3}\right)^2 - \left(\frac{y}{2}\right)^2 = \underbrace{1+k^2}_{\text{POSITIVE}}$



Asymptotes are at

$$\left(\frac{z}{3}\right)^2 - \left(\frac{y}{2}\right)^2 = 0$$

or $z = \pm\frac{3}{2}y$.

When $k=0$ have points $(0, \pm 3)$

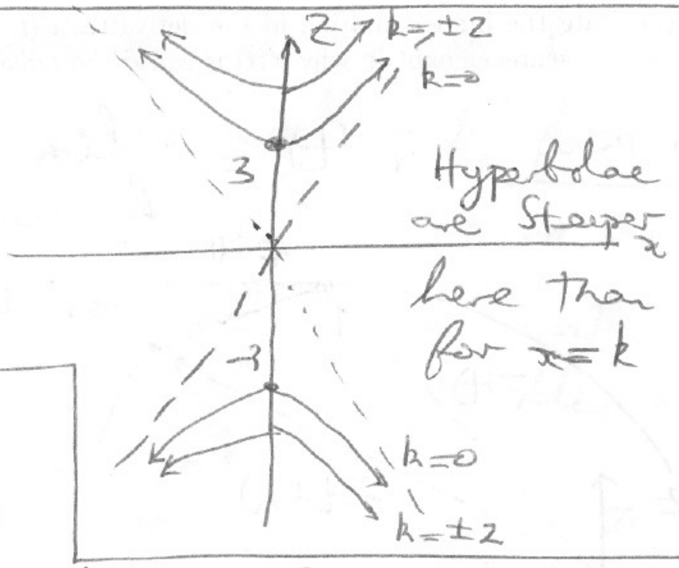
For larger k, traces ~~start at~~ INTERSECT z axis further from origin

$\underline{y=k}$ SIMILAR to $x=k$ but

$$\left(\frac{z}{3}\right)^2 - x^2 = 1 + \left(\frac{k}{2}\right)^2$$

Asymptotes at $z = \pm 3x$

When $k=0$ have $(0, \pm 3)$



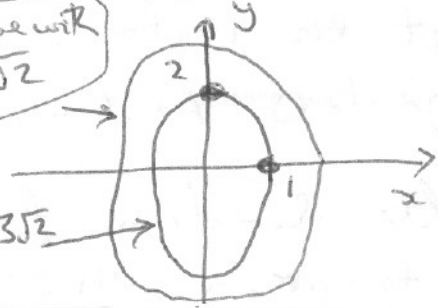
Hyperbolae are Steeper here than for $x=k$

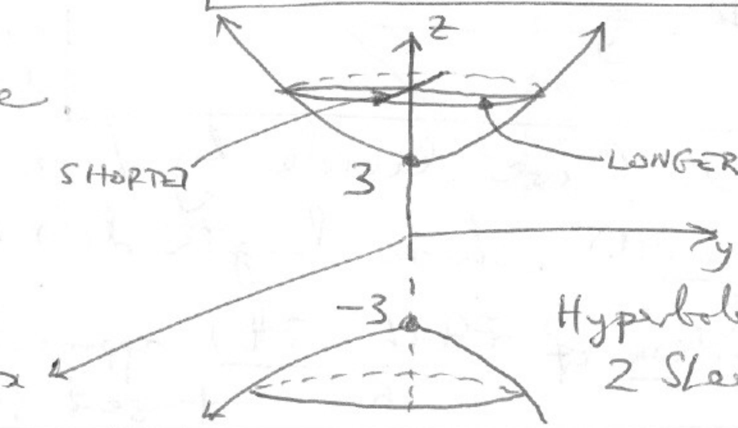
$\underline{z=k}$ $x^2 + \left(\frac{y}{2}\right)^2 = \left(\frac{k}{3}\right)^2 - 1$

IF $|k|<3$ NO trace as $\left(\frac{k}{3}\right)^2 - 1 < 0$

IF $k=3$ Trace is origin

IF $|k|>3$ Trace is ellipse.





Hyperboloid of 2 Sheets

(4) [15 pts]

(a) Sketch the image in the xy-plane of the parametrized curve $\mathbf{r}(t) = (3\cos t, 4\sin t)$, where $0 \le t \le \pi$.

$x = 3\cos t,\ y = 4\sin t$

So ~~to~~ eliminate t use

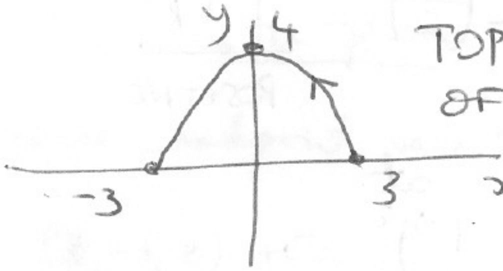
$1 = \cos^2 t + \sin^2 t$

$1 = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2$

Image curve is on an ELLIPSE.

$\vec{r}(0) = (3,0)$, $\vec{r}\left(\frac{\pi}{2}\right) = (0,4)$,

$\vec{r}(\pi) = (0,-3)$

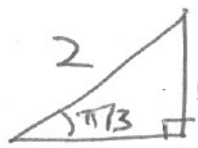


TOP HALF OF ELLIPSE as $0 \le t \le \pi$

(b) Calculate $\mathbf{r}'(\pi/3)$, where $\mathbf{r}$ is the parametrized curve in (a).

$\vec{r}'(t) = (-3\sin t, 4\cos t)$

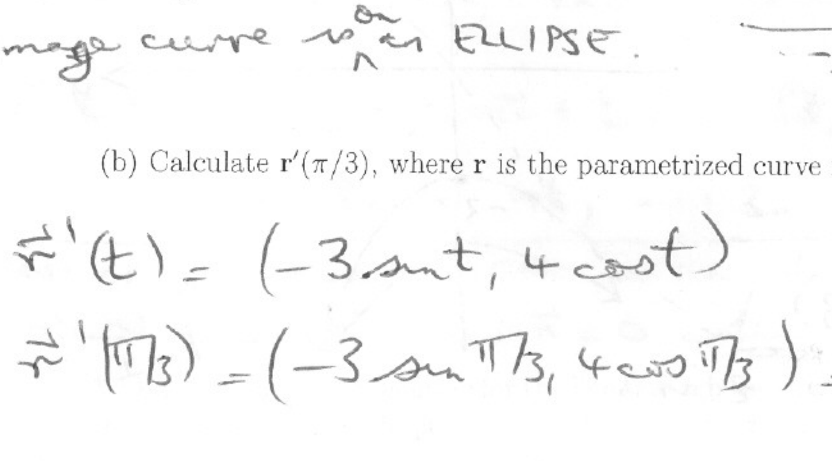
$\vec{r}'(\pi/3) = (-3\sin \pi/3, 4\cos \pi/3) = \left(-\frac{3\sqrt{3}}{2}, 2\right)$



(c) State the limit definition of the derivative $\mathbf{r}'(t)$ of a parametrized curve $\mathbf{r}$. Using a picture and an English sentence explain why $\mathbf{r}'(t)$ is called the *tangent vector* to the curve at $\mathbf{r}(t)$.

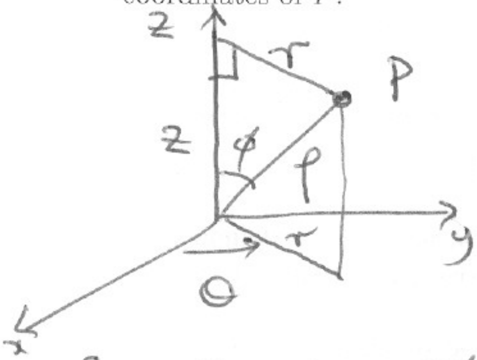
DEFINITION

$$\vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

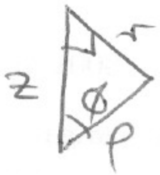


Since $\vec{r}(t+h)$ and $\vec{r}(t)$ are vectors $\vec{r}(t+h) - \vec{r}(t)$ is also a vector. From the picture we see this is a SECANT vector to curve. As $h \to 0$ the length of the secant goes to 0 but if we divide by h we get the vector $\frac{\vec{r}(t+h) - \vec{r}(t)}{h}$ whose length does not go to 0. As $h \to 0$ the direction of $\frac{\vec{r}(t+h) - \vec{r}(t)}{h}$ approaches the direction of a tangent vector to curve. So $\vec{r}'(t)$ IS TANGENT VECTOR

(5) [12 pts]
(a) Suppose the spherical coordinates of a point P are $(\rho, \theta, \phi) = (4, \pi/6, \pi/3)$. Find the cylindrical coordinates of P.



Extract the right triangle

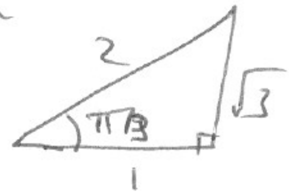


from picture on left

So $z = \rho \cos\phi$, $r = \rho \sin\phi$.

Cylindrical coords of P are (r, θ, z) with

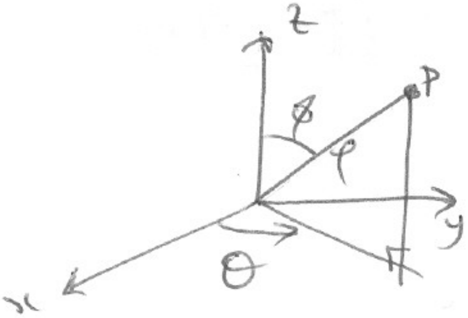
$r = \rho \sin\phi = 4 \sin \pi/3 = 2\sqrt{3}$



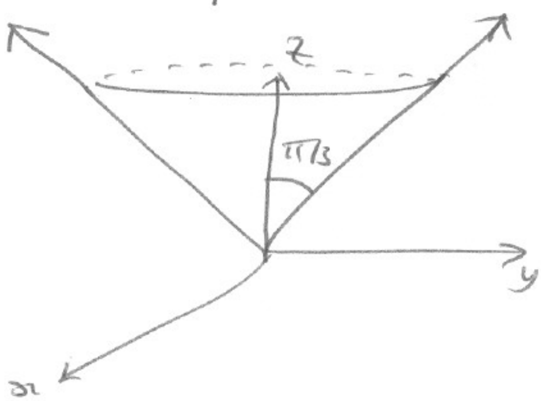
$\theta = \theta = \pi/6$

$z = \rho \cos\phi = 4 \cos \pi/3 = 2$ $(r, \theta, z) = (2\sqrt{3}, \pi/6, 2)$

(b) Sketch and describe in words the surface whose equation in spherical coordinates is $\phi = \pi/3$.



The set of all points P whose drop angle, ϕ, from the positive z-axis is $\phi = \pi/3$ is an up-facing cone with vertex at the origin. (On this cone ρ and θ can take on any value)



Pledge: *I have neither given nor received aid on this exam*

Signature: ____________________